11

ROTATIONAL VECTORS AND ANGULAR MOMENTUM

EXERCISES

Section 11.1 Angular Velocity and Acceleration Vectors

13. **INTERPRET** This problem is an exercise in determining the direction and magnitude of the angular velocity vector. From the direction and speed at which the car is traveling, we are to deduce the angular velocity of its wheels.

DEVELOP From Chapter 10 (Equation 10.3), we know that the magnitude of the angular velocity (i.e., the angular speed) is given by $\omega = v_{cm}r$. For this problem, we have v_{cm} = (70 km/h)(10^3 m/km)(1 h/3600 s) = 19.44 m/s and $r = d/2$ = (0.62 m)/2 = 0.31 m. The direction of the angular velocity vector can be determined using the right-hand rule (see Figure 11.1).

EVALUATE Inserting the given quantities into Equation 10.3 gives an angular speed of

$$\omega = v_{cm}/r = (19.44\ \text{m/s})/(0.31\ \text{m}) = 63\ \text{s}^{-1}$$

to two significant figures. If the car is rolling noth, the right-hand rule determines that the direction of the anular velocity vector is to the left, which is west. Therefore $\vec{\omega} = 63\ \text{s}^{-1}$ west .

ASSESS Notice that the angular speed may be reported in units of rad/s, but since radians are a dimensionless quantity, they are often left out, leaving s^{-1}, which is a frequency (Hz).

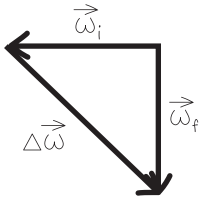
14. **INTERPRET** The problem asks us to determine the angular acceleration of the wheels of a car traveling north with a speed of 70 km/h and that makes a 90° left turn that lasts for 25 s.

DEVELOP The speed of the car is v_{cm} = 70 km/h = 19.44 m/s. Assuming that the wheels are rolling without slipping, the magnitude of the initial angular velocity is

$$\omega = \frac{v_{cm}}{r} = \frac{19.44\ \text{m/s}}{0.31\ \text{m}} = 62.7\ \text{s}^{-1}$$

With the car going north, the axis of rotation of the wheels is east-west. Since the top of a wheel is going in the same direction as the car, the right-hand rule gives the direction of $\vec{\omega}_i$ as west. In unit-vector notation, we write $\vec{\omega}_i = -\omega\hat{i}$.

After making a left turn, the angular speed remains unchanged, but the direction of $\vec{\omega}_f$ is now south (see sketch). In unit-vector notation, we write $\vec{\omega}_f = -\omega\hat{j}$.



EVALUATE Using Equation 11.1, we find the angular acceleration to be

$$\vec{\alpha}_{\text{ave}} = \frac{\Delta\vec{\omega}}{\Delta t} = \frac{\vec{\omega}_f - \vec{\omega}_i}{\Delta t} = \frac{-\omega\hat{j} - \left(-\omega\hat{i}\right)}{\Delta t} = \frac{\omega}{\Delta t}\left(\hat{i} - \hat{j}\right)$$
$$= \frac{62.7\ \text{s}^{-1}}{25\ \text{s}}\left(\hat{i} - \hat{j}\right) = \left(2.5\ \text{s}^{-2}\right)\left(\hat{i} - \hat{j}\right)$$

The magnitude of $\vec{\alpha}_{\text{ave}}$ is

$$|\vec{\alpha}_{\text{ave}}| = \frac{\sqrt{2}\omega}{\Delta t} = \frac{\sqrt{2}\left(62.7\ \text{rad/s}\right)}{25\ \text{s}} = 3.6\ \text{rad/s}^2$$

and $\vec{\alpha}_{\text{ave}}$ points in the south-east direction (in the direction of the vector $\hat{i} - \hat{j}$).

ASSESS Angular acceleration $\vec{\alpha}_{\text{ave}}$ points in the same direction as $\Delta\vec{\omega}$. The units can be reported as either rad/s^2 or s^{-2}.

15. **INTERPRET** This problem involves calculating the magnitude of the average acceleration given the initial and final angular velocities, and the time interval between the two. We are also asked to find the angle that the average angular acceleration vector makes with the horizontal.

DEVELOP Let the x axis be the horizontal direction (positive to the right), and the upward direction be the y axis. The the average angular acceleration vector is simply the difference between the final and initial angular velocities divided by the time interval between these two speeds (i.e., Equation 11.1). The initial angular velocity is $\omega_{\text{i}} = (45\ \text{rpm})\,\hat{j}$, the final angular speed is $\omega_{\text{f}} = (60\ \text{rpm})\,\hat{i}$, and the time interval is t = 15 s. To find the angle θ the average angular acceleration vector makes with the horizontal, use the fact that that $\tan\theta = \bar{\alpha}_y / \bar{\alpha}_x$.

EVALUATE (a) Inserting the given quantities into Equation 11.1, we find

$$\bar{\alpha} = \frac{\omega_{\text{f}} - \omega_{\text{i}}}{\Delta t} = \frac{(60\ \text{rpm})\,\hat{i} - (45\ \text{rpm})\,\hat{j}}{0.25\ \text{min}^{-1}} = \left(240\ \text{min}^{-2}\right)\hat{i} - \left(180\ \text{min}^{-2}\right)\hat{j} = \left(86{,}400\ \text{s}^{-2}\right)\hat{i} - \left(64{,}800\ \text{s}^{-2}\right)\hat{j}$$

The magnitude of the average acceleration is thus $\bar{\alpha} = \sqrt{\left(86{,}400\ \text{s}^{-2}\right)^2 + \left(64{,}800\ \text{s}^{-2}\right)^2} = 1.1\times10^6\ \text{s}^{-2}$ to two significant figures.

(b) The angle of the average angular acceleration vector with respect to the horiztonal is

$$\theta = \text{atan}\left(\frac{\bar{\alpha}_y}{\bar{\alpha}_x}\right) = \text{atan}\left(\frac{-64{,}800\ \text{s}^{-2}}{86{,}400\ \text{s}^{-2}}\right) = -37^\circ$$

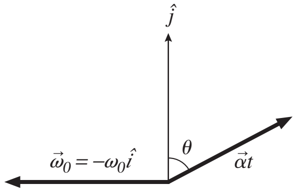
ASSESS Note that the quantities used to calculate part (b) were intermediate quantities, so more significant figures are retained. The final result, however, is reported to two significant figures, which reflects the precision of the data.

16. **INTERPRET** The problem involves angular velocity and angular acceleration. We are given the initial angular velocity of a wheel and asked to find its final angular velocity after an angular acceleration has been applied over a given time interval.

DEVELOP Draw a diagram of the situation with the initial vectors (see figure below). Take the x-axis east and the y-axis north, with positive angles measured CCW from the x-axis. In unit-vector notation, the initial angular velocity ω_i and the angular acceleration $\bar{\alpha}$ can be expressed as

$$\omega_{\text{i}} = \omega_{\text{i}}\hat{i} = (140\ \text{rad/s})\,\hat{i}$$
$$\bar{\alpha} = \alpha\left(\cos\theta_\alpha\hat{i} + \sin\theta_\alpha\hat{j}\right) = \left(35\ \text{rad/s}^2\right)\left[\cos\left(90^\circ + 68^\circ\right)\hat{i} + \sin\left(90^\circ + 68^\circ\right)\hat{j}\right]$$
$$= \left(-32.45\ \text{rad/s}^2\right)\hat{i} + \left(13.11\ \text{rad/s}^2\right)\hat{j}$$

The final angular velocity can be found by using Equation 11.1.



EVALUATE Using Equation 11.1, the angular velocity at t = 5.0 s is

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_f - \omega_i}{\Delta t}$$

$$\omega_f = \omega_i + \bar{\alpha}t = (140\text{ rad/s})\hat{i} + \left[\left(-32.45\text{ rad/s}^2\right)\hat{i} + \left(13.11\text{ rad/s}^2\right)\hat{j}\right](5.0\text{ s})$$
$$= (-22\text{ rad/s})\hat{i} + (66\text{ rad/s})\hat{j}$$

to two significant figures. The magnitude and direction of $\vec{\omega}_f$ are

$$\omega_f = \sqrt{(-22.3\text{ rad/s})^2 + (65.6\text{ rad/s})^2} = 69\text{ rad/s}$$

and

$$\theta_f = \tan^{-1}\left(\frac{\omega_{f,y}}{\omega_{f,x}}\right) = \tan^{-1}\left(\frac{-22.3\text{ rad/s}}{65.6\text{ rad/s}}\right) = -19°$$

or 19° west of north.

ASSESS Because the x component of the angular acceleration is negative, $\Delta\omega_x$ is also negative. On the other hand, a positive α_y yields $\Delta\omega_y > 0$.

Section 11.2 Torque and the Vector Cross Product

17. **INTERPRET** This problem involves finding the torque about the origin given a force and the position vector that indicates where the force is applied.

DEVELOP Use Equation 11.2 to find the torque. The position vector r is $\vec{r} = (3\text{ m})\hat{i} + (1\text{ m})\hat{j}$.

EVALUATE (a) For a force $\vec{F} = (12\text{ N})\hat{i}$, the torque is

$$\vec{\tau} = \vec{r}\times\vec{F} = \left[(3\text{ m})\hat{i} + (1\text{ m})\hat{j}\right]\times(12\ \text{ N}\cdot\text{m})\hat{i} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3\text{ m} & 1\text{ m} & 0\text{ m} \\ 12\text{ N} & 0\text{ N} & 0\text{ N}\end{vmatrix} = (-12\text{ N}\cdot\text{m})\hat{k}$$

(b) For a force $\vec{F} = (12\text{ N})\hat{j}$

$$\vec{\tau} = \vec{r}\times\vec{F} = \left[(3\text{ m})\hat{i} + (1\text{ m})\hat{j}\right]\times(12\ \text{ N}\cdot\text{m})\hat{j} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3\text{ m} & 1\text{ m} & 0\text{ m} \\ 0\text{ N} & 12\text{ N} & 0\text{ N}\end{vmatrix} = (36\text{ N}\cdot\text{m})\hat{k}$$

(c) For a force $\vec{F} = (12\text{ N})\hat{k}$

$$\vec{\tau} = \vec{r}\times\vec{F} = \left[(3\text{ m})\hat{i} + (1\text{ m})\hat{j}\right]\times(12\ \text{ N}\cdot\text{m})\hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3\text{ m} & 1\text{ m} & 0\text{ m} \\ 0\text{ N} & 0\text{ N} & 12\text{ N}\end{vmatrix} = (12\text{ N}\cdot\text{m})\hat{i} + (36\text{ N}\cdot\text{m})\hat{j}$$

ASSESS For part (c), the magnitude is $\tau = \sqrt{(12\text{ N}\cdot\text{m})^2 + (36\text{ N}\cdot\text{m})^2} = 38\text{ N}\cdot\text{m}$ and the direction is $\theta = \text{atan}(36\text{ N}\cdot\text{m}/12\text{ N}\cdot\text{m}) = 72^\circ$ counter clockwise from the x axis and in the x-y plane.

18. **INTERPRET** We are asked to find the torque about two different points produced by an applied force. The problem is about taking a cross product.

DEVELOP The torque vector is defined as $\vec{\tau} = \vec{r} \times \vec{F}$, where $\vec{F}$ is the force vector and $\vec{r}$ is the position vector which points from the axis of rotation to the point where the force is acting. The direction of $\vec{\tau}$ is determined by the right-hand-rule.

EVALUATE **(a)** For this part, $\vec{r} = (3\text{ m})\hat{i}$. Therefore, with $\vec{F} = (1.3\text{ N})\hat{i} + (2.7\text{ N})\hat{j}$, the torque is

$$\vec{\tau} = \vec{r} \times \vec{F} = (3\text{ m})\hat{i} \times \left[(1.3\text{ N})\hat{i} + (2.7\text{ N})\hat{j}\right] = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3\text{ m} & 0\text{ m} & 0\text{ m} \\ 1.3\text{ N} & 2.7\text{ N} & 0\text{ N} \end{vmatrix} = (8.1\text{ N}\cdot\text{m})\hat{k}$$

(b) Here we have $\vec{r} = ((3\text{ m}))\hat{i} - \left[(-1.3\text{ m})\hat{i} + (2.4\text{ m})\hat{j}\right] = (4.3\text{ m})\hat{i} - (2.4\text{ m})\hat{j}$. Therefore, the torque is

$$\vec{\tau} = \vec{r} \times \vec{F} = \left[(4.3\text{ m})\hat{i} - (2.4\text{ m})\hat{j}\right] \times \left[(1.3\text{ N})\hat{i} + (2.7\text{ N})\hat{j}\right] = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4.3\text{ m} & -2.4\text{ m} & 0\text{ m} \\ 1.3\text{ N} & 2.7\text{ N} & 0\text{ N} \end{vmatrix}$$

$$= (11.6\text{ N}\cdot\text{m})\hat{k} + (3.1\text{ N}\cdot\text{m})\hat{k} = (15\text{ N}\cdot\text{m})\hat{k}$$

ASSESS The torque vector $\vec{\tau}$ is perpendicular to both $\vec{r}$ and $\vec{F}$. It points in the direction normal to the plane formed by $\vec{r}$ and $\vec{F}$.

19. **INTERPRET** You want to know what torque is supplied by the deltoid muscle about the shoulder joint when your arm is outstretched.

DEVELOP From Equation 11.2, the torque is $\vec{\tau} = \vec{r} \times \vec{F}$, with the magnitude equaling $rF \sin\theta$.

EVALUATE The distance between the shoulder joint (i.e., where the arm pivots) and where the deltoid force is applied is given as $r = 18$ cm. The angle between the corresponding radial vector and the muscle force is $\theta = 180^\circ - 15^\circ = 165^\circ$. The magnitude of the torque is then

$$\tau = rF \sin\theta = (0.18\text{ m})(67\text{ N})\sin 165^\circ = 3.1\text{ N}\cdot\text{m}$$

By the right-hand rule, we start with our fingers pointing to the right in the direction of $\vec{r}$, and then rotate them upwards in the direction of $\vec{F}$. Our thumb points up, so the torque of $3.1\text{ N}\cdot\text{m}$ points out of the page.

ASSESS Is this enough torque to keep the arm outstretched? Let's assume the arm has a mass of about 3 kg (corresponding to a weight of about 30 N), and its center of mass is 30 cm from the shoulder joint. The gravitational force will pull the arm down at 90° to the horizontal arm direction, thus generating a torque in the opposite direction with a magnitude of $\tau \simeq (30\text{ N})(0.3\text{ m}) = 9\text{ N}\cdot\text{m}$. Therefore, the deltoid muscle would need help from other muscles to keep the arm horizontal.

Section 11.3 Angular Momentum

20. **INTERPRET** This problem involves a dimensional analysis of angular momentum. We are to express angular momentum in terms of the fundamental SI units, in terms of Newtons, and in terms of Joules.

DEVELOP Angular momentum is given by Equation 11.3, $\vec{L} = \vec{r} \times \vec{p}$. Given that *r* has units of distance (m) and *p* has units of mass times velocity, we can find the SI units of angular momentum. From Newton's second law (for constant mass) $F = ma$, we see that force is the product of mass and acceleration, so the units a newton are kg·m·s^{-2}. Simiarly, energy (J) can be expressed as a force multiplied by distance (consider work, $W = \vec{F} \cdot \Delta\vec{r}$, Equation 6.5), so the SI units of a the joule are kg·m^2·s^{-2}.

EVALUATE (a). Using the dimensions of linear momentum (= kg·m·s^{-1}), the units of angular momentum are (kg·m·s^{-1})(m) = kg·m^2·s^{-1}.
(b) Because the units of a newton are kg·m·s^{-2}, angular momentum can be expressed as kg·m^2·s^{-1} = N·m·s.
(c) Because energy (J) can be expressed as force times distance, we have J = N·m, so the units of angular momentum are kg·m^2·s^{-1} = (N·m)·s = J·s.

ASSESS From Equation 11.2, we see that torque has units of N·m, so a torque multiplied by a time gives an angular momentum. This is just the definition of an angular impluse.

21. **INTERPRET** This problem asks us to find the angular momentum of a ball given its linear velocity, its mass, and the distance from its axis of rotatoin.

DEVELOP The angular momentum of an object about a point is defined as (see Equation 11.3)

$$L = \vec{r} \times \vec{p}$$

where $\vec{p}$ is the linear momentum and $\vec{r}$ is the position vector of the object relative to that point. We may also express $\vec{L}$ as

$$\vec{L} = \vec{r} \times \vec{p} = (rp\sin\theta)\,\hat{n}$$

where θ is the angle between $\vec{r}$ and $\vec{p}$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{r}$ and $\vec{p}$. For this problem, we can assume that the ball is traveling in a circle of radius r and speed v. Since the velocity of the ball, $\vec{v}$, is perpendicular to $\vec{r}$, the magnitude of the angular momentum about the center is $L = |\vec{r} \times \vec{p}| = rp = rmv$.

EVALUATE From the problem statement, we have r = 1.2m + 0.9 m = 2.1 m and v = 27 m/s. Therefore,

$$L = rmv = (2.1 \text{ m})(7.3 \text{ kg})(27 \text{ m/s}) = 4.1 \text{ kg}\cdot\text{m}^2\cdot\text{s}^{-1}$$

ASSESS The direction of $\vec{L}$ is parallel to the axis of rotation. It is perpendicular to both $\vec{v}$ and $\vec{r}$.

22. **INTERPRET** For this problem, we are to find the angular speed given the angular momentum and the rotational inertia of an object.

DEVELOP Use Equation 11.4, $\vec{L} = I\vec{\omega}$ to find the angular speed of the gymnast.

EVALUATE Given that $L = 470\text{ kg}\cdot\text{m}^2\cdot\text{s}^{-1}$ and the $I = 62\text{ kg}\cdot\text{m}^2$, the angular speed of the gymnast must be

$$\omega = \frac{L}{I} = \frac{470 \text{ kg}\cdot\text{m}^2\cdot\text{s}^{-1}}{62 \text{ kg}\cdot\text{m}^2} = 7.6 \text{ s}^{-1}$$

ASSESS The angular speed has units of frequency, as expected. It may equivalently be expressed as 7.6 rad/s, because radians are dimensionless.

23. **INTERPRET** We are given the elements of rotational inertia and the angular velocity of the hoop and are to find the corresponding angular momentum. We will need to use Table 10.2 to find the rotational inertia.

DEVELOP For an object rotating about a fixed axis, its angular momentum can be expressed as (see Equation 11.4) $\vec{L} = I\vec{\omega}$, where I is the moment of inertia of the object, and $\vec{\omega}$ is its angular velocity about its axis. From Table 10.2, we find that the rotational inertia of a hoop rotating about its axis is $I = mr^2$.

EVALUATE With ω = 170 rpm = 17.89 rad/s, the magnitude of $\vec{L}$ is

$$L = I\omega = mr^2\omega = (0.64 \text{ kg})(0.45 \text{ m})^2(17.8 \text{ rad/s}) = 2.3 \text{ J}\cdot\text{s}$$

The direction of $\vec{L}$ is along the axis of rotation according to the right-hand rule.

ASSESS The angular momentum vector $\vec{L}$ points in the same direction as $\vec{\omega}$.

24. **INTERPRET** The problem asks for the angular momentum of a spinning baseball.

DEVELOP Equation 11.4 gives the angular momentum as $\vec{L} = I\vec{\omega}$. In this case, we are only concerned with the magnitude of the baseball's angular momentum. We are told to treat the ball as a uniform solid sphere spinning about an axis through its center, in which case its rotational inertia is given by $I = \frac{2}{5}MR^2$ (from Table 10.1).

EVALUATE Taking care to convert the rotational speed to rad/s, the angular momentum is

$$L = \tfrac{2}{5}MR^2\omega = \tfrac{2}{5}(0.145 \text{ kg})\left(\tfrac{1}{2}0.074 \text{ m}\right)^2(2000 \text{ rpm})\left[\frac{2\pi \text{ rad/s}}{60 \text{ rpm}}\right] = 1.7\times10^{-2} \text{ J}\cdot\text{s}$$

ASSESS The value seems reasonable. The units are correct, since $\text{kg}\cdot\text{m}^2/\text{s} = (\text{kg}\cdot\text{m}^2/\text{s}^2)\cdot\text{s} = \text{J}\cdot\text{s}$.

Section 11.4 Conservation of Angular Momentum

25. **INTERPRET** This problem involves conservation of angular momentum, which we can use to find the angular speed of a spinning wheel after a piece of clay is dropped onto it and sticks to its surface.

DEVELOP If the clay is dropped vertically onto a horizontally spinning wheel, the angular momentum about the vertical spin axis is conserved. Conservation of angular momentum is expressed as

$$\vec{L}_\text{i} = \vec{L}_\text{f} \quad \Rightarrow \quad I_\text{i}\vec{\omega}_\text{i} = I_\text{f}\vec{\omega}_\text{f}$$

For this problem, the direction of the angular velocity does not change, so this expression for conservation of angular momentum reduces to its scalar form, $I_\text{i}\omega_\text{i} = I_\text{f}\omega_\text{f}$. The intial rotational inertia is $I_\text{i} = I_\text{wheel} = 6.40$ kg·m^2·s^{-1}, and the final rotational inertia is $I_\text{f} = I_\text{wheel} + m_\text{clay}r^2$.

EVALUATE Inserting the given quantities into the expression from conservation of angular momentum, the final angular velocity is

$$\omega_\text{f} = \frac{I_\text{i}}{I_\text{f}}\omega_\text{i} = \left(\frac{I_\text{wheel}}{I_\text{wheel} + m_\text{clay}r^2}\right)\omega_i = \frac{6.40\text{ kg·m}^2}{6.40\text{ kg·m}^2 + (2.70\text{ kg})(0.460\text{ m})^2}(19.0\text{ rpm}) = 17.4\text{ rpm}$$

ASSESS The clay increases the total rotational inertia of the system, so the angular speed decreases, as required by conservation of angular momentum.

26. **INTERPRET** This problem involves conservation of angular momentum and the work-energy theorem. The former we can use to find the angular speed of the merry-go-round after the children sit on it, and the latter we can use to find the energy lost in the transaction.

DEVELOP Conservation of angular momentum demands that $I_\text{i}\vec{\omega}_\text{i} = I_\text{f}\vec{\omega}_\text{f}$. The initial rotational inertia is $I_\text{i} = 120$ kg·m^2, the initial angular velocity is $\omega_\text{i} = 0.50$ rev/s. The final rotational inertia is $I_\text{f} = I_\text{i} + 4m_\text{c}r^2$, where m_c is the mass of one child. To find the energy lost when the children jump onto the merry-go-round, consider the work-energy theorem Equation 6.14, $\Delta K = W_\text{net} = \vec{f}_\text{k}\cdot\Delta\vec{r}$, where f_k is the force due to friction, which acts parallel to $\Delta\vec{r}$. From the rotational version of the work-energy theorem, Equaiton 10.19, we see that we can find the change in kinetic energy using the result of part (a).

EVALUATE (a) From conservation of momentum, we have

$$I_\text{i}\omega_\text{i} = I_\text{f}\omega_\text{f}$$

$$\omega_\text{f} = \omega_\text{i}\frac{I_\text{i}}{I_\text{f}} = \omega_\text{i}\frac{I_\text{i}}{I_\text{i} + 4m_\text{c}r^2} = (0.50\text{ rev/s})\frac{120\text{ kg}\cdot\text{m}^2}{120\text{ kg}\cdot\text{m}^2 + 4(25\text{ kg})(3.0\text{ m})^2}$$

$$= (0.174\text{ rev/s})\left(\frac{2\pi\text{ rad}}{\text{rev}}\right) = 1.09\ \text{ rad/s}$$

(b) Using $\omega_\text{i} = (0.50\text{ rev/s})(2\pi\text{ rad/rev}) = \pi\text{ rad/s}$, the change in kinetic energy is $\Delta K = K_\text{f} - K_\text{i}$, which gives

$$\Delta K = \frac{1}{2}I_\text{f}\omega_\text{f}^2 - \frac{1}{2}I_\text{i}\omega_\text{i}^2 = \frac{\left(I_\text{i} + 4m_\text{c}r^2\right)\omega_\text{f}^2 - I_\text{i}\omega_\text{i}^2}{2}$$

$$= \frac{\left[120\text{ kg}\cdot\text{m}^2 + 4(25\text{ kg})(3.0\text{ m})^2\right](1.09\text{ rad/s})^2 - \left(120\text{ kg}\cdot\text{m}^2\right)(\pi\text{ rad/s})^2}{2}$$

$$= 386\text{ J}$$

ASSESS We could also find the energy lost using the fact that $\Delta K = I\omega^2 = L^2/I$. This gives

$$\Delta K = \frac{1}{2}\left(\frac{1}{I_\text{f}} - \frac{1}{I_\text{i}}\right)L^2$$

$$= \frac{1}{2}\left(\frac{1}{345\text{ kg}\cdot\text{m}^2} - \frac{1}{120\text{ kg}\cdot\text{m}^2}\right)\left(120\text{ kg}\cdot\text{m}^2 \times \pi\text{ s}^{-1}\right)^2 = 386\text{ J}$$

where we have used the fact that angular momentum is conserved so $L_\text{i} = L_\text{f} = L$.

27. **INTERPRET** In this problem we are asked about the period of a star formed by a collapsing cloud. We can use conservation of angular momentum to find the answer.

DEVELOP If we assume there are no external torques and no mass loss during the collapse of the star-forming cloud, its angular momentum is conserved, so $I_i\omega_i = I_f\omega_f$ The initial and final rotational inertias may be found from Table 10.2, which gives $I_i = 2mr_i^2/5$ and $I_f = 2mr_f^2/5$. From this,we can solve for the final period of the star using $T_f = 2\pi/\omega_f$.

EVALUATE Given that the mass involved does not change,conservation of angular momentum gives

$$I_i\omega_i = I_f\omega_f$$
$$\frac{2}{5}MR_i^2\omega_i = \frac{2}{5}MR_f^2\omega_f$$
$$\frac{\omega_i}{\omega_f} = \left(\frac{R_f}{R_i}\right)^2$$

Thus, the final period is

$$T_f = T_i\left(\frac{R_f}{R_i}\right)^2 = \left(1.4\times10^6 \text{ y}\right)\left(\frac{7.0\times10^8 \text{ m}}{1.0\times10^{13} \text{ m}}\right)^2 = 6.86\times10^{-3} \text{ y} = 2.5 \text{ days}$$

ASSESS In current models of star formation, the collapsing cloud does not maintain a spherical shape, forming a flattened disk instead, and the central star retains just a fraction of the original cloud's mass.

28. **INTERPRET** This problem involves a skater holding two weights in his hands. His rotational speed will change when he brings the weights to his chest.

DEVELOP If the skater is twirling on frictionless horizontal ice, his angular momentum about the vertical rotation axis is conserved: $I_i\omega_i = I_f\,\omega_f$. When the arms are initially outstretched, the weights contribute to the rotational inertia: $I_i = I_{s,out} + 2MR^2$. Here, $I_{s,out}$ is the skater's rotational inertia when his arms are outstretched, and R is the distance the weights are from the rotational axis. When the arms are brought into the chest, we assume they no longer contribute to the rotational inertia, since the distance for the axis goes to zero. The final rotational inertia is just that of the skater with arms to his chest: $I_f = I_{s,in}$.

EVALUATE Solving for the final rotational speed gives:

$$\omega_f = \frac{I_i}{I_f}\omega_i = \frac{5.7 \text{ kg}\cdot\text{m}^2 + 2(2.5 \text{ kg})(0.76 \text{ m})^2}{4.2 \text{ kg}\cdot\text{m}^2}(3.0 \text{ rev/s}) = 6.1 \text{ rev/s}$$

ASSESS The skater doubles his speed, which seems reasonable. If he didn't have the weights in his hand, his final rotational speed would be 4.1 rev/s, which is only a 30% increase over the initial speed.

PROBLEMS

29. **INTERPRET** This problem is an exercise in calculating torque, given the force and the position relative to an axis at which the force is applied.

DEVELOP Use Equation 11.2, $\vec{\tau} = \vec{r}\times\vec{F}$ to calculate the torque, given that $\vec{r} = (18 \text{ cm})\hat{i} + (5.5 \text{ cm})\hat{j}$ and. $\vec{F}(88 \text{ N})\hat{i} - (23)\hat{j}$.

EVALUATE Evaluating the cross product gives

$$\vec{\tau} = \left[(18 \text{ cm})\hat{i} + (5.5 \text{ cm})\hat{j}\right]\times\left[\vec{F}(88 \text{ N})\hat{i} - (23)\hat{j}\right] = \det\begin{pmatrix}\hat{i} & \hat{j} & \hat{k}\\ 18 \text{ cm} & 5.5 \text{ cm} & 0 \text{ cm}\\ 88 \text{ N} & -23 \text{ N} & 0 \text{ N}\end{pmatrix} = (-9.0 \text{ N}\cdot\text{m})\hat{k}$$

ASSESS Thus the torque is the direction defined by the bolt.

30. **INTERPRET** The problem is an exercise in vector multiplication (cross product). It asks us to find the direction of a vector $\vec{B}$, given the directions of another vector $\vec{A}$ and their cross product $\vec{A}\times\vec{B}$.

DEVELOP The cross product, $\vec{A}\times\vec{B}$, is perpendicular to the plane defined by $\vec{A}$ and $\vec{B}$. We are given that $\vec{A}\times\vec{B} = -A^2\hat{k}$ and these vectors lie in the *x*-*y* plane. For simplicity, let's write the two vectors as

$$\vec{A} = A\left(\cos\theta_A\hat{i} + \sin\theta_A\hat{j}\right)$$
$$\vec{B} = B\left(\cos\theta_B\hat{i} + \sin\theta_B\hat{j}\right)$$

where $\theta_A = 30°$ and θ_B are measured counterclockwise from the *x* axis. Using the above expressions, the cross product $\vec{A}\times\vec{B}$ is

$$\vec{A}\times\vec{B} = AB\left(\cos\theta_A\hat{i} + \sin\theta_A\hat{j}\right)\times\left(\cos\theta_B\hat{i} + \sin\theta_B\hat{j}\right) = AB\left(\cos\theta_A\sin\theta_B - \sin\theta_A\cos\theta_B\right)\hat{k}$$
$$= -AB\sin\left(\theta_A - \theta_B\right)\hat{k}$$

Using the information given in the problem statement, the angle θ_B can be calculated.

EVALUATE The problem states that $\vec{A}\times\vec{B} = -A^2\hat{k}$. The right-hand rule implies that the angle between $\vec{A}$ and $\vec{B}$, measured clockwise from $\vec{A}$, is less than 180°; namely, $\theta_A - \theta_B < 180°$ or $-150° < \theta_B < 30° = \theta_A$.
The magnitude of $\vec{A}\times\vec{B}$ is $AB\sin\left(\theta_A - \theta_B\right) = 2A^2\sin\left(\theta_A - \theta_B\right) = A^2$ (as given, with $B = 2A$), so

$$\sin\left(\theta_A - \theta_B\right) = \frac{1}{2}$$

or $\theta_A - \theta_B = 30°$ or 150°. When this is combined with the given value of θ_A and the range of θ_B, one finds that $\theta_B = 0°$ or $-120°$ (i.e., along the *x*-axis or 120° clockwise from the *x*-axis).

ASSESS The vector corresponding to $\theta_B = 0°$ can be written as $\vec{B}_1 = B\hat{i} = 2A\hat{i}$. Similarly, for $\theta_B = -120°$, we have

$$\vec{B}_2 = B\left[\cos(-120°)\hat{i} + \sin(-120°)\hat{j}\right] = 2A\left[(-1/2)\hat{i} - \left(\sqrt{3}/2\right)\hat{j}\right] = -A\hat{i} - \sqrt{3}A\hat{j}$$

With $\vec{A} = A\left[\cos(30°)\hat{i} + \sin(30°)\hat{j}\right] = A\left[\left(\sqrt{3}/2\right)\hat{i} + (1/2)\hat{j}\right]$ the cross products are

$$\vec{A}\times\vec{B}_1 = A\left[\left(\sqrt{3}/2\right)\hat{i} + (1/2)\hat{j}\right]\times(2A)\hat{i} = -A^2\hat{k}$$

and

$$\vec{A}\times\vec{B}_2 = A\left[\left(\sqrt{3}/2\right)\hat{i} + (1/2)\hat{j}\right]\times\left[-A\hat{i} - \sqrt{3}A\hat{j}\right] = \left(-\frac{3}{2}A^2 + \frac{1}{2}A^2\right)\hat{k} = -A^2\hat{k}$$

Both results indeed agree with the condition given in the problem statement.

31. **INTERPRET** We're asked to calculate the torque exerted by the ball player in order to bring the baseball to rest.

DEVELOP The player exerts a torque around his shoulder, which results in a stopping force on the ball. From Equation 11.2, the average torque is $\bar{\tau} = r\bar{F}_{\text{stop}}$, where we have taken into account that the vertically-held arm and the horizontally-directed force are at right angles, so $\sin\theta = 1$. We won't worry about the direction of the torque, just the magnitude. The average stopping force is equal to $\bar{F}_{\text{stop}} = m\bar{a}$, where the average acceleration can be found through Equation 2.11: $\bar{a} = v_0^2/2\Delta x$. Here, v_0 is the initial speed, and Δx is the stopping distance. We have neglected the negative sign because we're only looking for magnitudes.

EVALUATE The average torque exerted by the player on the ball is:

$$\bar{\tau} = \frac{rmv_0^2}{2\Delta x} = \frac{(63\text{ cm})(0.145\text{ kg})(42\text{ m/s})^2}{2(5.00\text{ cm})} = 1600\text{ N}\cdot\text{m}$$

ASSESS One can arrive at the answer by using Equation 10.11: $\bar{\tau} = I\bar{\alpha}$. In this case, the rotational inertia is that of the ball rotating around the shoulder joint: $I = mr^2$. The average angular acceleration relates to the ball's average linear acceleration through Equation 10.5: $\bar{\alpha} = \bar{a}/r$, so the final expression is the same: $\bar{\tau} = rm\bar{a}$.

32. **INTERPRET** In this problem we are asked to verify the vector identity $\vec{A}\cdot(\vec{A}\times\vec{B}) = 0$.

DEVELOP The key to the proof is to realize that the cross product $\vec{A}\times\vec{B}$ is perpendicular to $\vec{A}$ and $\vec{B}$.

EVALUATE Let $\vec{C} = \vec{A} \times \vec{B}$. If $\vec{C}$ is perpendicular to $\vec{A}$ and $\vec{B}$, then their scalar products must vanish:

$$\vec{A} \cdot \vec{C} = \vec{A} \cdot \left(\vec{A} \times \vec{B}\right) = 0$$
$$\vec{B} \cdot \vec{C} = \vec{B} \cdot \left(\vec{A} \times \vec{B}\right) = 0$$

(Recall that $\vec{A} \cdot \vec{B} = AB\cos\theta$, where θ is the angle between $\vec{A}$ and $\vec{B}$.)

ASSESS An alternative approach is to use the component forms. Let's write the vectors as

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}, \quad \vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

The cross product $\vec{A} \times \vec{B}$ is

$$\begin{aligned}\vec{A} \times \vec{B} &= \left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right)\left(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}\right) \\ &= A_xB_x\hat{i}\times\hat{i} + A_xB_y\hat{i}\times\hat{j} + A_xB_z\hat{i}\times\hat{k} + A_yB_x\hat{j}\times\hat{i} + A_yB_y\hat{j}\times\hat{j} \\ &\quad + A_yB_z\hat{j}\times\hat{k} + A_zB_x\hat{k}\times\hat{i} + A_zB_y\hat{k}\times\hat{j} + A_zB_z\hat{k}\times\hat{k} \\ &= \left(A_yB_z - A_zB_y\right)\hat{i} + \left(A_zB_x - A_xB_z\right)\hat{j} + \left(A_xB_y - A_yB_x\right)\hat{k}\end{aligned}$$

The dot product $\vec{A} \cdot (\vec{A} \times \vec{B})$ then becomes

$$\begin{aligned}\vec{A} \cdot \left(\vec{A} \times \vec{B}\right) &= A_x\left(A_yB_z - A_zB_y\right) + A_y\left(A_zB_x - A_xB_z\right) + A_z\left(A_xB_y - A_yB_x\right) \\ &= \left(A_yA_z - A_zA_y\right)B_x + \left(A_zA_x - A_xA_z\right)B_y + \left(A_xA_y - A_yA_x\right)B_z \\ &= \left(\vec{A} \times \vec{A}\right) \cdot \vec{B} = 0\end{aligned}$$

In general, $\vec{A} \cdot \left(\vec{B} \times \vec{C}\right)$ is called the triple scalar product and $\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) = \left(\vec{A} \times \vec{B}\right) \cdot \vec{C}$, i.e., the "dot" and the "cross" in the triple scalar product can be interchanged. This is equivalent to a cyclic permutation of the three vectors,

$$\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) = \vec{C} \cdot \left(\vec{A} \times \vec{B}\right) = \vec{B} \cdot \left(\vec{C} \times \vec{A}\right)$$

On the other hand, interchanging any two vectors introduces a minus sign,

$$\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) = -\vec{C} \cdot \left(\vec{B} \times \vec{A}\right) = -\vec{B} \cdot \left(\vec{A} \times \vec{C}\right) = -\vec{A} \cdot \left(\vec{C} \times \vec{B}\right)$$

33. **INTERPRET** This problem involves calculating the angular momentum of an object. We are given the mass distribution of the object, so we can find its rotational inertia, and we also know its angular velocity.

DEVELOP Use Equation 11.4, $\vec{L} = I\vec{\omega}$, to compute the angular momentum. The rotational inertia of the weights and bar about the specified axis is (see Table 10.2)

$$I = 2m_{\text{wt}}\left(\frac{L}{2}\right)^2 + \frac{1}{12}m_{\text{bar}}L^2$$

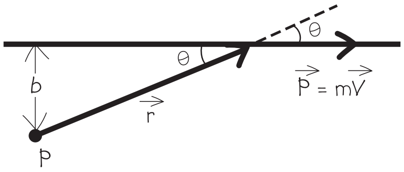
EVALUATE With $\omega = 10.0 \text{ rpm} = 1.05 \text{ rad/s}$, the angular momentum about this axis is

$$L = I\omega = \left[2(25 \text{ kg})(0.8 \text{ m})^2 + \frac{1}{12}(15 \text{ kg})(1.6 \text{ m})^2\right](1.05 \text{ rad/s}) = 37 \text{ J}\cdot\text{s}$$

ASSESS The greater the angular speed, the larger the angular momentum.

34. **INTERPRET** This problem involves calculating the angular momentum of an object of mass m traveling at speed v along a straight line. The point about which the angular momentum is to be calculated is a point a perpendicular distance b from the straight line. We are to show that the angular momentum is mvb, regardless of the position of the object on the line.

DEVELOP Apply Equation 11.3, $\vec{L} = \vec{r} \times \vec{p}$, using the geometry as drawn in the sketch below. Note that $\vec{p} = m\vec{v}$ and $b = r\sin\theta$.



EVALUATE Evaluating the cross product, we find $L = |\vec{r} \times \vec{p}| = pr\sin\theta = mvb$.

ASSESS The direction of $\vec{L}$ is also the same, for any position along the trajectory (in this case, into the page as sketched).

35. **INTERPRET** We need to find the angular momentum of a disk-shaped rotor that is part of a micromechanical device that measures blood flow.

DEVELOP The angular momentum of the rotor is $L = I\omega$, where the rotational inertia is that of a disk: $I = \frac{1}{2}MR^2$. We don't explicitly know the rotor's mass, but the material is silicon, which has a density of $\rho = 2.33\ \text{g/cm}^3$.

EVALUATE The mass of the rotor is the density times the volume: $M = \rho\left(d \cdot \pi R^2\right)$, where d is the rotor's thickness. The radius is half the diameter: $R = 150\mu m$, and the 800-rpm rotational speed converted to SI units is: $\omega = 83.8$ rad/s. So the angular momentum of the rotor during the tests is

$$L = I\omega = \frac{\pi}{2}\rho d R^4 \omega$$
$$= \frac{\pi}{2}\left(2.33\times10^3\ \text{kg/m}^3\right)\left(2.0\times10^{-6}\ \text{m}\right)\left(150\times10^{-6}\ \text{m}\right)^4\left(83.8\ \text{rad/s}\right) = 3.1\times10^{-16}\ \text{J}\cdot\text{s}$$

ASSESS This is a very small angular momentum, but we expect it to be. Otherwise, the device would significantly disturb the blood flow it is designed to measure.

36. **INTERPRET** This problem asks us to find the angular momentum of an object about a given point. To do so, we will need to calculate the rotational inertia of the object, given its rotational inertia about its center of mass (i.e., the rotational inertia if it were to rotate about an axis that goes through its center of mass). We are given the new axis about which the object rotates, so we can apply the parallel-axis theorem to find the rotational inertia about this new axis. The second part of the problem involves the rotational analog of Newton's second law, which we can use to find the torque required to achieve the given angular momentum in the given time.

DEVELOP Use Equation 11.4 , $\vec{L} = I\vec{\omega}$, to find the angular momentum of the bat about point P. Applying the parallel-axis theorem (see Equation 10.17) to the bat gives us the rotational inertia about an axis through the point P as $I_p = Mh^2 + I_{cm}$, with $I_{cm} = 0.048\ \text{kg}\cdot\text{m}^2$ and $h = 43$ cm. The angular velocity can be found using Equaiton 10.3, $v = r\omega$, where r is the distance from the P; $r = 43\ \text{cm} + 31\ \text{cm} = 74$ cm and $v = 50$ m/s. The direction of ω can be found using the right-hand rule, so if the bat is swung counter clockwise, the angular velocity vector is oriented out of the page, and if it is swung clockwise, the angular velocity vector is oriented into the page. To find the torque needed, apply Equation 11.5 in discreet form: $\Delta L/\Delta t = \tau$.

EVALUATE (a) Inserting the given quantities into Equation 11.4 gives

$$L = I\omega = \left(Mh^2 + I_{cm}\right)\frac{v}{r} = \left[\left(0.88\ \text{kg}\right)\left(0.43\ \text{m}\right)^2 + 0.048\ \text{kg}\cdot\text{m}^2\right]\left(\frac{50\ \text{m/s}}{0.74\ \text{m}}\right) = 14\ \text{J}\cdot\text{s}$$

The direction of the angular momentum is either out or into the plane of the page, depending on whether the bat is rotated counter clockwise or clockwise, respectively.

(b) From the rotational analog of Newton's second law, the torque needed to acheive this angular momentum in 0.25 s is

$$\tau = \frac{\Delta L}{\Delta t} = \frac{L_f - \overbrace{L_i}^{=0}}{\Delta t} = \frac{L_f}{\Delta t} = \frac{14.2\ \text{J·s}}{0.25\ \text{s}} = 57\ \text{N}\cdot\text{m}$$

The direction of the torque is the same as that of the angular momentum because the initial momentum was zero.

ASSESS In ft-lbs, the torque is

$$\tau = (57\ \text{N}\cdot\text{m})\left(\frac{1\ \text{lb}}{4.448\ \text{N}}\right)\left(\frac{1\ \text{ft}}{0.3048\ \text{m}}\right) = 42\ \text{ft·lb}$$

which is a reasonable result (i.e., possible for a human to achieve).

37. **INTERPRET** This problem asks us to calculate the rotational inertia of a tire if the design reduces the angular momentum by a certain percentage, while keeping the linear speed fixed.

DEVELOP The linear speed of the car is related to its angular speed as $v = \omega r$ (see Equation 10.3). Keeping v fixed implies

$$\omega_1 r_1 = \omega_2 r_2$$

From Equation 11.4, $L = I\omega$, the new rotational inertia can be computed.

EVALUATE The new specifications require that

$$\frac{L_2}{L_1} = \frac{I_2\omega_2}{I_1\omega_1} = 0.7 \quad \Rightarrow \quad \frac{I_2}{I_1} = 0.7\frac{\omega_1}{\omega_2}$$

Using $\omega_1 = \omega_2 r_2/r_1$, we obtain

$$I_2 = (0.70)I_1\frac{\omega_1}{\omega_2} = (0.70)I_1\frac{R_2}{R_1} = (0.70)(0.32\ \text{kg}\cdot\text{m}^2)\left(\frac{35\ \text{cm}}{38\ \text{cm}}\right) = 0.21\ \text{kg}\cdot\text{m}^2$$

ASSESS The general condition is

$$\frac{L_2}{L_1} = \frac{I_2\omega_2}{I_1\omega_1} = \frac{I_2R_1}{I_1R_2} \quad \Rightarrow \quad L_2 = \left(\frac{I_2}{I_1}\right)\left(\frac{R_1}{R_2}\right)L_1$$

A decrease in angular momentum ($L_2 < L_1$) can be achieved by either decreasing r_1/r_2 or I_2/I_1. In our problem, the ratio r_1/r_2 = (38 cm)/(35 cm) = 1.09 actually is increased. However, this change is accompanied by a greater decrease in rotational inertia I_2/I_1 = (0.206 kg·m^2)/ (0.32 kg·m^2) = 0.64.

38. **INTERPRET** This problem involves conservation of angular momentum, which we can use to find the angular speed when the mouse is at the center of the turntable. The second part involves the work-energy theorem, which we can use to find the work done by the mouse. The mouse does work when it exerts reaction forces to friction between its feet and the turntable.

DEVELOP Apply conservation of angular momentum. Because the axis of rotation does not change, we can use the scalar form, so $L_f = L_i$. The final angular momentum is $L_f = I_t\omega_f$ and the initial angular momentum is $L_i = I_t\omega_i + mr^2$, where m = 19.5 g is the mass of the mouse and r = 25 cm is its distance from the axis of rotation. With the final angular velocity known, we can apply the work-energy theorem Equation 10.19,

$$W = \Delta K_{rot} = I_f\omega_f^2/2 - I_i\omega_i^2/2$$

to find the work done by the mouse.

EVALUATE (**a**) Inserting the given values into the expression for conservation of angular momentum gives

$$\omega_f = \omega_i I_i/I_f = (22.0\ \text{rpm})\frac{\left[0.0154\ \text{kg}\cdot\text{m}^2 + (0.0195\ \text{kg})(0.25\ \text{m})^2\right]}{0.0154\ \text{kg·m}^2} = 23.7\ \text{rpm}.$$

(**b**) From the work-energy theorem, the work done by the mouse is

$$W = K_\text{f} - K_\text{i} = \frac{1}{2}I_\text{f}\omega_\text{f}^2 - \frac{1}{2}I_\text{i}\omega_\text{i}^2 = \frac{1}{2}I_\text{f}\omega_\text{f}^2\left(1-\omega_\text{i}/\omega_\text{f}\right)$$

$$= \frac{1}{2}\left(0.0154\text{ kg}\cdot\text{m}^2\right)\left(23.7\text{ rev/min}\right)^2\left(\frac{2\pi\text{ rad}}{\text{rev}}\right)^2\left(\frac{1\text{ min}}{60\text{ s}}\right)^2\left(1-\frac{22.0\text{ rev/min}}{23.7\text{ rev/min}}\right) = 3.49\text{ mJ}.$$

where we used conservation of angular momentum from part (a).

ASSESS The units or rev/min had to be changed to rad/s in part (b) because rev/min are not SI units.

39. **INTERPRET** This problem involves conservation of angular momentum, which we can use to calculate the motion of the dog relative to the ground.

DEVELOP Walking once around relative to the turntable, the dog describes an angular displacement of $\Delta\theta_\text{D}$ relative to the ground, and the turntable one of $\Delta\theta_\text{T}$ in the opposite direction, such that $\Delta\theta_\text{D} - \Delta\theta_\text{T} = 2\pi.$ The vertical component of the angular momentum of the dog-and-turntable system is conserved (which was zero initially), so

$$L_i = L_f \quad\Rightarrow\quad 0 = I_\text{D}\omega_\text{D} + I_\text{T}\omega_\text{T} = I_\text{D}\left(\frac{\Delta\theta_\text{D}}{\Delta t}\right) - I_\text{T}\frac{\Delta\theta_\text{T}}{\Delta t}$$

where the angular velocities (which are in opposite directions) have been rewritten in terms of the angular displacements and the common time interval. The rotational inertias about the axis of rotation are

$$I_\text{D} = mR^2 = (17\text{ kg})(1.81\text{ m})^2 = 55.7\text{ kg}\cdot\text{m}^2$$

and $I_\text{T} = 95\text{ kg}\cdot\text{m}^2.$ These results allow us to solve for $\Delta\theta_\text{D}$.

EVALUATE Eliminating $\Delta\theta_\text{T}$, we find

$$0 = I_\text{D}\Delta\theta_\text{D} - I_\text{T}\left(2\pi - \Delta\theta_\text{D}\right) = \left(I_\text{D} + I_\text{T}\right)\Delta\theta_\text{D} - 2\pi I_\text{T}$$

or

$$\frac{\Delta\theta_\text{D}}{2\pi} = \frac{I_\text{T}}{I_\text{D} + I_\text{T}} = \frac{95\text{ kg}\cdot\text{m}^2}{55.7\text{ kg}\cdot\text{m}^2 + 95\text{ kg}\cdot\text{m}^2} = 0.63$$

In other words, $\Delta\theta_\text{D}$ is 63% of a full circle relative to the ground.

ASSESS We find that $\Delta\theta_\text{D}$, the angular displacement relative to the ground, decreases with I_D. This is what we expect from conservation of angular momentum.

40. **INTERPRET** This problem involves conservation of momentum and the work-energy theorem. The former can be used to find the student's mass given the rotational parameters of the turntable on which she is standing and the wheel that she is holding, and the latter can be used to find the work she does in turning the wheel upside down.

DEVELOP Because the turntable is frictionless, there are no external torques about its axis, and the z component of angular momentum is conserved. The initial angular momentum is just that due to the spinning wheel,

$$L_\text{i} = I_\text{W}\omega_\text{W}$$

When the wheel is inverted, the student and turntable acquire an angular momentum

$$L_\text{f} = L_\text{T} + L_\text{S} - L_\text{W} = \left(I_\text{T} + \overbrace{m_\text{W}h^2}^{\text{neglect}}\right)\omega_\text{T} + I_\text{S}\omega_\text{S} - I_\text{W}\omega_\text{W}$$

$$= I_\text{T}\omega_\text{T} + I_\text{S}\omega_\text{S} - I_\text{W}\omega_\text{W}$$

where we have neglected the rotational inertia due to the center of mass of the wheel, and we have subtracted the angular momentum of the wheel because it is not oriented in the opposite (i.e., downward) direction. We know that $\omega_\text{T} = \omega_\text{S} = 70$ rpm, and that $\omega_\text{W} = 130$ rpm. The rotational inertias are $I_\text{T} = 0.31\text{ kg}\cdot\text{m}^2$, $I_\text{S} = m_\text{S}r^2/2$ with $r = 0.30$ m, and $I_W = 0.22\ \text{kg}\cdot\text{m}^2$, so we can solve for the student's mass m_S. Apply the rotational version of the work-energy theorem (Equation 10.19) to find the work done reversing the wheel. This gives

$$W = \Delta K_{\text{rot}} = \frac{1}{2} I_{\text{tot}} \omega_f^2 - \frac{1}{2} I_{\text{tot}} \overbrace{\omega_i^2}^{=0} = I_{\text{tot}} \omega_i^2$$

where the initial angular velocity of the entire system is zero (note that the kinetic energy of the wheel does not change in this experiment) and the final angular velocity is $\omega_f = \omega_T = 70$ rpm. The total rotational inertia of the system is $I_{\text{tot}} = I_T + I_S$ (we are neglecting the rotational inertia due to the center of mass of the wheel).

EVALUATE (**a**) Equating the initial and final angular momenta, we find

$$I_W \omega_W = I_T \omega_T + I_S \omega_S - I_W \omega_W$$

$$I_S = \frac{2 I_W \omega_W - I_T \omega_T}{\omega_S} = \frac{m_S R^2}{2}$$

$$m_S = 2\left(\frac{2 I_W \omega_W - I_T \omega_T}{R^2 \omega_S}\right) = 2\left(\frac{2\left(0.22\ \text{kg}\cdot\text{m}^2\right)(130\ \text{rpm}) - \left(0.31\ \text{kg}\cdot\text{m}^2\right)(70\ \text{rpm})}{(0.15\ \text{m})^2 (70\ \text{rpm})}\right) = 45\ \text{kg}$$

(**b**) Evaluating the expression above for work gives

$$W = \frac{1}{2} I_{\text{tot}} \omega_f^2 = \frac{1}{2}\left(I_T + I_S\right)\omega_f^2 = \frac{1}{2}\left(I_T + m_S R^2/2\right)$$

$$= \frac{1}{2}\left(0.31\ \text{kg}\cdot\text{m}^2 + (45.1\ \text{kg})(0.15\ \text{m})^2/2\right)\left(70\ \frac{\text{rev}}{\text{min}}\right)^2 \left(\frac{2\pi\ \text{rad}}{\text{rev}}\right)^2 \left(\frac{1\ \text{min}}{60\ \text{s}}\right)^2 = 22\ \text{J}$$

ASSESS From the expression for work, we see that a student with a larger radius would have to do more work to flip the wheel, which is reasonable because the student's rotational inertia would be greater, so it would require more work to get him to rotate.

41. **INTERPRET** This problem is about the rotational motion of the skaters, given their initial linear speed and radius of the circle they traverse. The aim is to keep the final linear speed and centripetal force below the stated maximums. The key concept here is conservation of angular momentum.

DEVELOP If the ice is frictionless, the only external force on the skaters is the force that brings the end-skater to a sudden stop at a point we'll call *P*. (Note: The forces they exert on each other through their hands are internal forces.) The stopping force exerts no torque about point *P*, so the total angular momentum about a vertical axis through *P* is conserved. Initially, the other seven skaters are each moving with the same linear momentum $(p = mv_0)$ in a direction perpendicular to the line that connects them $(\sin\theta = 1)$. So from Equation 11.3, the angular momentum of each skater about *P* is

$$L_{0n} = |\vec{r}_n \times \vec{p}_n| = r_n (mv_0)\sin\theta = mv_0 r_n$$

where r_n is the distance between the *n*-th skater and the point *P*: $r_n = n(\ell/7)$ for $n = 1, 2, \ldots, 7$ and $\ell = 12$ m. The total initial angular momentum is the sum $L_0 = \sum_{n=1}^{7} L_{0n}$, which will be conserved when the group starts rotating and has an angular momentum of $L_f = I\omega$. Here, the rotational inertia is $I = \sum_{n=1}^{7} mr_n^2$. From all this we can determine the rotational speed, which will give us the linear speed and centripetal force on the outside skater $(n = 7)$.

EVALUATE The total initial angular momentum is

$$L_0 = \sum_{n=1}^{7} mv_0 r_n = \frac{mv_0\ell}{7}\sum_{n=1}^{7} n = \frac{mv_0\ell}{7}\left[\frac{7\times 8}{2}\right] = 4mv_0\ell$$

where we have used $\sum_{n=1}^{N} n = N(N+1)/2$. Similarly, the rotational inertia of the 7 skaters around point *P* is

$$I = \sum_{n=1}^{7} mr_n^2 = \frac{m\ell^2}{49}\sum_{n=1}^{7} n^2 = \frac{m\ell^2}{49}\left[\frac{7\times 8\times 15}{6}\right] = \frac{20m\ell^2}{7}$$

where we have used $\sum_{n=1}^{N} n^2 = N(N+1)(2N+1)/6.$ Since angular momentum is conserved $(L_0 = L_f)$, we can solve for the angular speed:

$$\omega = \frac{L_0}{I} = \frac{4mv_0\ell}{\frac{20}{7}m\ell^2} = \frac{7v_0}{5\ell}$$

The outside skater will have a tangential speed of $v = \omega\ell,$ so in order to keep this below 8.0 m/s, the initial speed can't exceed:

$$v_0 = \tfrac{5}{7}v < \tfrac{5}{7}(8.0\text{ m/s}) = 5.7\text{ m/s}$$

The force on the outside skater's hand is the centripetal force: $F = ma_c = m\ell\omega^2.$ To keep This below 300 N, the initial speed can't exceed:

$$v_0 = \frac{5}{7}\sqrt{\frac{F\ell}{m}} < \frac{5}{7}\sqrt{\frac{(300\text{ N})(12\text{ m})}{(60\text{ kg})}} = 5.5\text{ m/s}$$

This limit is stricter than the one above. The greatest speed that the skaters can go before the rotational maneuver is 5.5 m/s.

Assess Notice that the outside skater will be going 1.4 times faster following the maneuver. By contrast, the skaters closer to the point P will slow down after the maneuver $(v_n = \omega r_n)$. This makes sense: to keep the total angular momentum constant, some skaters will gain angular momentum, while others will lose it.

42. **Interpret** This problem is an exercise in vector multiplication. Given that the dot product of two vectors is twice the magnitude of their cross product, we are to find the angle between the two vectors.

Develop Expressed mathematically, the relationship between the two vectors is

$$\vec{A}\cdot\vec{B} = 2\left|\vec{A}\times\vec{B}\right|$$

Use the definitions of the dot and cross products (Chapters 6 and 11, respectively). The dot product is $\vec{A}\cdot\vec{B} = AB\cos\theta$ and the magnitude of the cross product is $\left|\vec{A}\times\vec{B}\right| = AB\sin\theta$. Use these relationships to find the angle θ between the vectors $\vec{A}$ and $\vec{B}$.

Evaluate Inserting the expressions for the dot and cross products into the given relationship and solving for θ gives

$$AB\cos\theta = 2AB\sin\theta \quad\Rightarrow\quad \theta = \text{atan}\left(\frac{1}{2}\right) = 26.6^\circ$$

Assess Notice that the angle used for both vector products is the same angle.

43. **Interpret** This problem involves conservation of angular momentum, which we can use to find the angular speed of the bird feeder after the bird lands on it. We will need to consider the inertia of the bird feeder, which is given, and that of the bird, which we will take as a point particle of mass mb = 140 g rotating at 19 cm from the axis. Since the bird and the feeder initially have opposite angular momenta with respect to the bird-feeder axis, it is possible that the direction of the feeder's angular momentum will change; so we will keep track of the direction by the sign of ω, which is the angular speed of the bird-feeder.

Develop Apply conservation of angular momentum. The initial angular momentum is the sum of that due to the bird feeder and that due to the bird. Mathematically, this is expressed as

$$L_i = L_{bf} + L_b = I_{bf}\omega_{bf} + I_b\omega_b$$

If we define the initial angular velocity of the bird feeder as the positive direction, the $\omega_{bf} = 5.6$ rpm, $\omega_b = -v_b/r_{bf}$, where we have introduced the negative sign because the bird's initial angular velocity is opposite to that of the bird feeder. The rotational inertia of the bird can be taken as $m_b r_{bf}^2$. The final angular momentum is

$$L_f = L_{bfb} = \left(I_{bf} + I_b\right)\omega_{bfb}$$

where the subscript bfb indicates the bird-feeder-bird combination. By conservation of angular momentum, we can equate L_i and L_f and solve for the final angular speed, ωβφβ.

EVALUATE Equating the initial and final angular momenta, we find

$$I_{bf}\omega_{bf} + \left(m_b r_{bf}^2\right)\left(-\frac{v_b}{r_{bf}}\right) = \omega_{bfb}\left(I_{bf} + m_b r_{bf}^2\right)$$

$$\begin{aligned}\omega_{bfb} &= \frac{I_{bf}\omega_{bf} - m_b v_b r_{bf}}{I_{bf} + m_b r_{bf}^2} \\ &= \frac{\left(0.12\text{ kg}\cdot\text{m}^2\right)(5.6\text{ rpm})(2\pi\text{ rad/rev})(1\text{ min}/60\text{ s}) - (0.14\text{ kg})(1.1\text{ m/s})(0.19\text{ m})}{\left(0.12\text{ kg}\cdot\text{m}^2\right) + (0.14\text{ kg})(0.19\text{ m})^2} \\ &= 0.329\text{ rad/s} = 3.1\text{ rpm}\end{aligned}$$

ASSESS The sign of the final angular speed is the same as the sign of the initial angular speed, so the bird feeder continues to rotate in the same direction, albeit at a slower speed. Thus, the angular momentum of the bird feeder decreases, because it has absorbed the oppositely directed angular momentum of the bird.

44. **INTERPRET** This problem involves finding the force that causes a given torque when applied at a given point. Because of the vector nature of the quantities involved, we can equate the different (i.e., the *x* and *y*) components of the vectors that are equated.

DEVELOP Apply Equation 11.2, $\vec{\tau} = \vec{r} \times \vec{F}$, and equate the different vector components to find the *y* component of the force. The position vector is $\vec{r} = (2.0\text{ m/s})\hat{i}$, the force is $\vec{F} = (3.1\text{ N})\hat{i} + F_y\hat{j} + F_z\hat{k}$, and the torque is $\vec{\tau} = (4.6\text{ N}\cdot\text{m})\hat{k}$.

EVALUATE Evaluating the cross product gives

$$\vec{\tau} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2.0\text{ m} & 0\text{ m} & 0\text{ m} \\ 3.1\text{ N} & F_y & F_z \end{vmatrix} = (2.0\text{ m})F_z\hat{j} + (2.0\text{ m})F_y\hat{k} = (4.6\text{ N}\cdot\text{m})\hat{k}$$

Equating the y components gives $F_y = (4.6\text{ N·m})/(2.0\text{ m}) = 2.3$ N and $F_z = 0$ N. Thus, the angle the force makes with the x axis is $\theta = \text{atan}(F_y/F_x) = \text{atan}(3.1\text{ N}/2.3\text{ N}) = 37°$.

ASSESS By the right-hand rule, we can see that with both the position vector and the force vector in the *x-y* plane, the torque will be in the *z* direction, so we could have set $F_z = 0$ from the beginning.

45. **INTERPRET** The problem is about the rotational motion of the turntable. Tossing a piece of clay onto its surface is like a totally inelastic collision from Section 9.5. In this case, the total angular momentum is conserved.

DEVELOP The forces that cause the clay to stick to the turntable are internal forces (i.e. between clay and turntable). There are no external forces that can generate a torque around the turntable's axis, so the angular momentum of the turntable/clay system in the vertical direction is conserved. If we take the sense of rotation of the turntable to define the positive direction of vertical angular momentum, then the system's initial angular momentum is

$$L_i = I\omega + mvd$$

where we assume here that the clay hits the turntable with the same direction that the table is turning. After the collision, the clay turns at the same speed as the table, so the final angular momentum is

$$L_f = I\omega_f + md^2\omega_f = \left(I + md^2\right)\omega_f$$

By conservation of angular momentum, the clay's initial velocity is equal to

$$v = d\omega_f + \frac{I\left(\omega_f - \omega\right)}{md}$$

EVALUATE (a) If $\omega_f = \frac{1}{2}\omega$, then the clay hits the table with speed:

$$v = \frac{d\omega}{2} + \frac{-I\omega}{2md} = d\omega\left(\tfrac{1}{2} - I/2md^2\right)$$

(b) If $\omega_f = \omega$, then the clay hits the table with speed: $v = d\omega$.

(c) If $\omega_f = 2\omega$, then the clay hits the table with speed:

$$v = 2d\omega + \frac{I\omega}{md} = d\omega\left(2 + I/md^2\right)$$

ASSESS We have written the clay velocity in terms of $d\omega$, which is the initial linear speed of the turntable at the radius where the clay hits. If the clay hits with $v < d\omega$, then the collision will slow down the turntable, but if $v > d\omega$, the turntable will speed up.

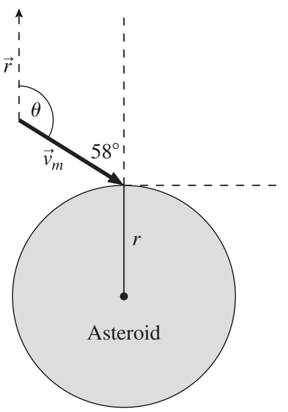
46. **INTERPRET** This problem involves conservation of momentum, which we can use to find the final angular momentum of the asteroid after it is impacted by the meteorite. Because the meteorite is traveling in the equatorial plane of the asteroid, its angular momentum is parallel to the asteroid's angular momentum (consider the right-hand rule). Therefore, we can dispense with the vector notation, and consider only angular momentum about the asteroid's axis.

DEVELOP Draw a diagram of the situation (see figure below). To use conservation of momentum, we must express the initial and final angular momenta. For the asteroid, because we are given the mass distribution (i.e., the rotational inertia) and the angular speed, we will use the scalar form of Equation 11.4 $L_a = I_a\omega_a$ to express its angular momentum. From Table 10.2, the rotational inertia of the asteroid is $I_a = 2Mr^2/5$, where M is the asteroid's mass. For the meteorite, we know its linear velocity and its position of impact, so we will use Equation 11.3, $L_m = rp_m \sin\theta$, where $p_m = mv_m$, where m is the meteorite's mass. The initial angular momentum of the asteroid-meteorite system is thus

$$L_i = L_a + L_m = I_a\omega_a + rp_m \sin\theta = \frac{2}{5}Mr^2\omega_a + rmv_m \sin\theta$$

where $\theta = 180° - 58° = 122°$. The final angular momentum is

$$L_f = \left(I_a + mr^2\right)\omega_f = \left(\frac{2}{5}Mr^2 + mr^2\right)\omega_f$$



EVALUATE Equating the initial and final angular momentum, we can solve for the final angular speed of the asteroid. The result is

$$\frac{2}{5}Mr^2\omega_a + rmv_m \sin\theta = \left(\frac{2}{5}Mr^2 + mr^2\right)\omega_f$$

$$m = \frac{2Mr\left(\omega_f - \omega_a\right)}{5\left(v_m \sin\theta - r\omega_f\right)}$$

Expressing angular velocity in terms of the period ($\omega = 2\pi/T$), we have

$$m = \frac{4\pi Mr\left(T_f^{-1} - T_a^{-1}\right)}{5\left(v_m \sin\theta - 2\pi r T_f^{-1}\right)} =$$

$$= \frac{4\pi\left(1.2\times10^{13}\text{ kg}\right)\left(1.0\text{ km}\right)\left[\left(3.9\text{ h}\right)^{-1} - \left(4.3\text{ h}\right)^{-1}\right]}{5\left[\left(8.4\text{ km/s}\right)\left(60\text{ s/1 h}\right)\sin\left(122^\circ\right) - 2\pi\left(1.0\text{ km}\right)\left(3.9\text{ h}\right)^{-1}\right]} = 1.7\times10^9\text{ kg}$$

ASSESS If the direction of rotation had been reversed by the impact, the meteoroid's mass would have been 20.5 times greater.

47. **INTERPRET** This problem asks us to calculate angular momenta given the mass distribution and rotational speeds (Appendix E) of the various planets of our solar system and the Sun. In particular, we are asked to estimate how much of the solar system's angular momentum about its center is associated with the Sun.

DEVELOP The planets orbit the Sun in planes approximately perpendicular to the Sun's rotation axis, so most of the angular momentum in the solar system is in this direction. We can estimate the orbital angular momentum of a planet by mvr, where m is its mass, v its average orbital speed, and r its mean distance from the Sun.

Compared to the orbital angular momentum of the four giant planets, everything else is negligible, except for the rotational angular momentum of the Sun itself, which can be estimated by assuming the Sun to be a uniform sphere rotating with an average period of $\frac{1}{2}(27+36)$ days. (The Sun's period of rotation at the surface varies from approximately 27 days at the equator to 36 days at the poles.)

EVALUATE The numerical data in Appendix E results in the following estimates:

Orbital Angular Momentum (mvr)		**%**
Jupiter	$19.2\times10^{42}\text{ J}\cdot\text{s}$	59.7
Saturn	$7.85\times10^{42}\text{ J}\cdot\text{s}$	24.4
Uranus	$1.69\times10^{42}\text{ J}\cdot\text{s}$	5.2
Neptune	$2.52\times10^{42}\text{ J}\cdot\text{s}$	7.8
Rotational Angular Momentum ($\frac{2}{5}MR^2\omega$)		
Sun	$0.89\times10^{42}\text{ J}\cdot\text{s}$	2.8
Total	$32.2\times10^{42}\text{ J}\cdot\text{s}$	99.9

ASSESS With $L_{orb} \gg L_{rot}$, we find that more than 97% of the total angular momentum of the solar system comes from the orbital angular momentum. In particular, the orbital motion of Jupiter alone accounts for roughly 60% of the total angular momentum.

48. **INTERPRET** To increase the surface area of this alien planet, you plan to hollow out its center. This will increase the rotational inertia, so to conserve angular momentum, the planet's rotation will slow down.

DEVELOP The planet is originally a solid sphere of radius R_0. When the planet is hollowed out, the radius of its outer surface is R, and the radius of its inner surface is $\frac{4}{5}R$, such that the shell thickness is $\frac{1}{5}R$. No material is added or taken away during this alteration, so the total mass, $M = \int dm$, should remain constant. To calculate the mass, divide the planet up into concentric shells of infinitesimal thickness. For a given shell of radius R', the mass is $dm = \rho\cdot 4\pi R'^2 dR'$, where ρ is the planet's density and $4\pi R'^2 dR'$ is the volume of the given shell. For the original planet, R' varies from 0 to R_0, while for the hollowed out planet, R' varies from $\frac{4}{5}R$ to R. Equating the mass integrals for the two cases gives:

$$\int_0^{R_0} 4\pi\rho R'^2 dR' = \int_{4R/5}^{R} 4\pi\rho R'^2 dR' \quad \rightarrow \quad R = \frac{5R_0}{\sqrt[3]{5^3 - 4^3}} \simeq 1.27R_0$$

This can be used to find the increase in surface area. But to find the change in the length of the day, you have to find the change in the rotational inertia of the planet. The original sphere has $I_0 = \frac{2}{5}MR_0^2$, from Table 10.1. However, the formula for a hollow sphere in Table 10.1, $I = \frac{2}{3}MR^2$, assumes the shell is thin, which is not the case here. What you can do is sum over the infinitesimal shells with mass dm defined above. Each of them has rotational inertia of:

$$dI = \frac{2}{3}(dm)R'^2 = \frac{8\pi}{3}\rho R'^4 dR' = \frac{2M}{R_0^3}R'^4 dR'$$

where the mass relation for the original sphere was used: $M = \rho\left(\frac{4\pi}{3}R_0^3\right)$. Integrating over all the shells in the hollowed sphere gives:

$$I = \int dI = \frac{2M}{R_0^3}\int_{4R/5}^{R} R'^4 dR' = \frac{2MR^5}{5R_0^3}\left[1 - \left(\tfrac{4}{5}\right)^5\right]$$

Notice that if R' varies from 0 to R_0, as in the original case, the integration returns the familiar result of $I = \frac{2}{5}MR_0^2$.

Evaluate By hollowing out the planet, the surface area increases by

$$\frac{A}{A_0} = \frac{4\pi R^2}{4\pi R_0^2} = \left(\frac{R}{R_0}\right)^2 = 1.27^2 = 1.61$$

The angular momentum is conserved, so $I_0\omega_0 = I\omega$. The period is inversely proportional to rotation speed $(T = 2\pi/\omega)$, so the length of the day will increase by

$$\frac{T}{T_0} = \frac{\omega_0}{\omega} = \frac{I}{I_0} = \frac{\frac{2}{5}MR^5/R_0^3}{\frac{2}{5}MR_0^2}\left[1 - \left(\tfrac{4}{5}\right)^5\right] = \left(\frac{R}{R_0}\right)^5\left[1 - \left(\tfrac{4}{5}\right)^5\right] = 2.22$$

Assess Suppose the hollowed sphere has thickness Δ, where $\Delta \ll R$. Then, the radii are related by $R \approx R_0/\sqrt[3]{3\Delta}$. Substituting this into the rotational inertia equation, and making a further approximation, gives

$$I = \frac{2MR^5}{5R_0^3}\left[1 - (1-\Delta)^5\right] \approx \frac{2}{5}MR^2\left(\frac{R}{R_0}\right)^3[5\Delta] = \frac{2}{3}MR^2$$

This is the formula for a hollow sphere given in Table 10.1, which shows that this expression only becomes valid when the thickness of the shell is much less than the radius.

49. **Interpret** This problem looks just like an inelastic collision, but instead of using conservation of linear momentum, we will use conservation of *angular* momentum. The angular momentum of each disk is in a single direction, so we can treat this as a one-dimensional problem.

Develop The masses of disk 1 and 2 are $m_1 = 440$ g and $m_2 = 270$ g, respectively. The radii are $r_1 = 0.035$ m and $r_2 = 0.23$ m. The initial angular speed of disk 1 is $\omega_1 = 180$ rpm. Use conservation of angular momentum, $L_i = L_f$, to find the final angular speed of both disks stuck together, and $\eta = 1 - K_f/K_i$, where $K = \frac{1}{2}I\omega^2$, to find the fraction of energy lost.

EVALUATE

(a) The initial angular momentum is

$$L_1 = I_1\omega_{1i} + I_2 \overbrace{\omega_{2i}}^{=0} = \frac{1}{2}m_1 r_1^2 \omega_{1i}.$$

The final angular momentum is

$$L_2 = \left(I_1 + I_2\right)\omega_f = \left(\frac{1}{2}m_1 r_1^2 + \frac{1}{2}m_2 r_2^2\right)\omega_f.$$

Conservation of angular momentum tells us that

$$L_1 = L_2$$
$$\frac{1}{2}m_1 r_1^2 \omega_{1i} = \left(\frac{1}{2}m_1 r_1^2 + \frac{1}{2}m_2 r_2^2\right)\omega_f$$
$$\omega_f = \omega_{1i}\left(\frac{m_1 r_1^2}{m_1 r_1^2 + m_2 r_2^2}\right) = (180\text{ rpm})\left(\frac{(440\text{ g})(3.5\text{ cm})^2}{(440\text{ g})(3.5\text{ cm})^2 + (270\text{ g})(2.3\text{ cm})^2}\right) = 140\text{ rpm}$$

(b) The initial kinetic energy is

$$K_i = \frac{1}{2}I_1\omega_{1i}^2 + \frac{1}{2}I_2 \overbrace{\omega_{2i}^2}^{=0} = \frac{1}{4}m_1 r_1^2 \omega_{1i}^2.$$

The final kinetic energy is

$$K_f = \frac{1}{2}\left(I_1 + I_2\right)\omega_f^2 = \frac{1}{4}\left(m_1 r_1^2 + m_2 r_2^2\right)\omega_f^2.$$

so the fraction of the initial kinetic energy lost to friction is

$$\eta = 1 - \frac{K_f}{K_i} = 1 - \frac{m_1 r_1^2 \omega_{1i}^2}{\left(m_1 r_1^2 + m_2 r_2^2\right)\omega_f^2} = 1 - \frac{\cancel{m_1 r_1^2}\ \cancel{\omega_{1i}^2}}{\cancel{\left(m_1 r_1^2 + m_2 r_2^2\right)}\left[\cancel{\omega_{1i}}\left(\frac{m_1 r_1^2}{m_1 r_1^2 + m_2 r_2^2}\right)\right]^{\cancel{2}}}$$
$$= 1 - \frac{m_1 r_1^2 + m_2 r_2^2}{m_1 r_1^2} = \frac{m_2 r_2^2}{m_1 r_1^2} = \frac{(270\text{ g})(2.3\text{ cm})^2}{(440\text{ g})(3.5\text{ cm})^2} = 0.265 = 27\ \%$$

to two significant figures.

ASSESS Note that the fractional energy loss doesn't depend on the initial energy. For this particular set of disks, 27% of the initial energy will be lost in the collision regardless of how fast the bottom disk is spinning!

50. **INTERPRET** This problem describes a rotational "explosion." Initially, there is zero angular momentum, with a spring compressed at the edge of a frictionless turntable. The spring is released and pushes a block off the turntable at an angle of 90° to the radial vector.

Two quantities are conserved: the kinetic energy and the angular momentum. We will use both in order to solve for the linear speed of the mass and the rotational speed of the turntable.

DEVELOP Conservation of energy tells us that the initial energy of the spring, $U_s = \frac{1}{2}kx^2$, is equal to the final combined kinetic energies of the mass and turntable, $K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$. Conservation of angular momentum tells us that the angular momentum is zero before and after the release of the spring, so $L = 0 = L_m + L_t$, where $L_m = |\vec{p}\times\vec{r}| = mvb.$ These two equations will allow us to solve for the two variables v and ω.

EVALUATE Conservation of energy gives us

$$\frac{1}{2}kx^2 = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$
$$kx^2 = mv^2 + I\omega^2.$$

Conservation of angular momentum gives us $0 = L_m + L_t = mvb + I\omega.$

(**a**) We solve the first equation for ω and substitute it into the second:

$$\omega = \pm\sqrt{\frac{kx^2 - mv^2}{I}}$$

$$mvb = -I\omega = \pm I\sqrt{\frac{kx^2 - mv^2}{I}}$$

$$(mvb)^2 = I^2\left(\frac{kx^2 - mv^2}{I}\right)$$

$$v^2\left(m^2b^2 + mI\right) = Ikx^2$$

$$v = \pm\sqrt{\frac{Ikx^2}{m^2b^2 + mI}}$$

Take this value for v and put it back into the equation for ω:

$$\omega = \pm\sqrt{\frac{kx^2 - mv^2}{I}} = \pm\sqrt{\frac{kx^2}{I} - \left(\frac{m}{I}\right)\left(\frac{Ikx^2}{m^2b^2 + mI}\right)} = \pm\sqrt{\frac{kx^2}{I} - \frac{kx^2}{mb^2 + I}} = \pm\sqrt{\frac{kx^2mb^2}{mb^2I + I^2}}$$

ASSESS The positive and negative signs indicate that the system is symmetric and could rotate counter clockwise or clockwise. We can check to see that this solution makes conceptual sense by letting the mass be very small compared to I. In that case, $v \to x\sqrt{k/m}$ and ω is zero, as we would expect. If b is zero, then ω is also zero and $v \to x\sqrt{k/m}$ again.

51. **INTERPRET** A solid spinning ball drops onto a frictional surface. At first it slides, but due to friction its spin will slow down and its linear speed will increase until it is purely rolling without sliding. We want to find the ball's angular speed when it begins purely rolling, and how long it takes.

DEVELOP From the problem statement, we see that the ball's mass is M, its radius is R, and its initial angular velocity around the horizontal axis is ω_0. The coefficient of kinetic friction between the ball and the surface is μ_k, so the frictional force is $F_f = \mu_s F_n = \mu Mg$. A torque acts on the ball due to the frictional force, which acts on the edge of the ball. This torque $\tau = -\mu_s MgR$ serves to slow the ball's rotation. Use $\tau = I\alpha$ to find the angular acceleration α and then use $\omega = \omega_0 + \alpha t$ to find the resulting angular speed. The frictional force on the ball also accelerates the ball, so we can use $F = Ma$ and $v = v_0 + at$ to find the speed of the ball. Combining this with the fact that the ball is no longer sliding when $R\omega = v$ allows us to find the time it takes to achieve rolling motion.

EVALUATE (**a**) The angular acceleration is

$$\alpha = \frac{\tau}{I} = \frac{-\mu_k MgR}{2MR^2/5} = -\frac{5\mu_k g}{2R}$$

where the negative sign comes from the fact that the frictional force always acts to counter the motion. Inserting this into the kinematic equation $\omega = \omega_0 + \alpha t$ gives

$$\omega = \omega_0 - \frac{5\mu_k g}{2R}t$$

Using the result from part (b) that $t = R\omega/\mu_k g$, we find that

$$\omega = \omega_0 - \frac{5\mu_k g}{2R}\left(\frac{R\omega}{\mu_k g}\right) = \omega_0 - \frac{5\omega}{2}$$

$$= \frac{2}{7}\omega_0$$

(**b**) The time it takes to achieve rolling motion is found from

$$a = \frac{F}{M} = \frac{\mu_k Mg}{M} = \mu_k g$$

so

$$v = v_0 + \mu_s gt$$

Inserting the condition $R\omega = v$ for rolling motion gives

$$R\omega = \mu_k gt$$

$$t = \frac{R\omega}{\mu_k g}$$

Using the result from part (a) that $\omega = 2\omega_0/7$, we find that

$$t = \frac{2R\omega_0}{\mu_k g}$$

ASSESS The answer to part **(a)** is surprising—it says that no matter what the size or speed of the ball, or the coefficient of friction, the angular speed of the ball when it stops sliding is 2/7 of its original value! However, the time it takes the ball to achieve rolling motion depends on the radius of the ball, its initial angular speed, and the coefficient of kinetic friction. Notice that the more slippery is the surface (i.e., for smaller μ_k), the longer it will take for the ball to achieve rolling motion, which is reasonable.

INTERPRET This problem

DEVELOP The speed

EVALUATE Using Equation

ASSESS Angular

52. **INTERPRET** This problem looks at a time-varying torque.

DEVELOP The torque and angular momentum are related by the rotational analog of Newton's second law: $\vec{\tau} = d\vec{L}/dt$. If we integrate the torque with respect to time, we obtain the angular momentum as a function of time. The given torque points in one direction, and consequently so does the angular momentum, so the vector notation can be dropped.

EVALUATE The angular momentum is

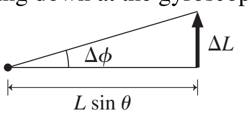
$$L = \int_0^t \tau dt' = \int_0^t \left(a + b\sin ct'\right) dt' = \left[at' - \frac{b}{c}\cos ct'\right]_0^t = at + \frac{b}{c}\left(1 - \cos ct\right)$$

ASSESS We are told that the object is initially stationary, so we can verify that indeed $L(t = 0) = 0$.

53. **INTERPRET** We're asked to derive the precession rate for a spinning gyroscope.

DEVELOP The torque is due to gravity. From Equation 11.2, it has a magnitude of $\tau = rF_g \sin\theta$, where θ is the angle between $\vec{L}$ and the vertical line extending up from the point where the gyroscope touches the bottom support. By the right-hand rule, the torque points in the direction perpendicular to the plane defined by $\vec{r}$ and the vertical.

EVALUATE Over a short time interval, Δt, the angular momentum changes in the direction given by the torque: $\Delta\vec{L} = \vec{\tau}\cdot\Delta t$, as shown in Figure 11.9. This change in $\vec{L}$ corresponds to a change in the rotational axis, since $\vec{L} = I\vec{\omega}$. We can characterize how the axis moves with a small angle $\Delta\phi = \Delta L / L\sin\theta$, as defined in the figure below. The view here is from above looking down at the gyroscope.



After the axis moves, the torque points in a new direction, but always in the direction perpendicular to the plane defined by $\vec{r}$ and the vertical. This leads to circular motion with a rotational speed of

$$\omega_p = \frac{\Delta\phi}{\Delta t} = \frac{1}{\Delta t}\left(\frac{\Delta L}{L\sin\theta}\right) = \frac{\tau}{L\sin\theta} = \frac{mgr\sin\theta}{L\sin\theta} = \frac{mgr}{L}$$

ASSESS This says the precession speed will be faster if the gyroscope has a larger mass and/or a longer radial length. It also says that the rate is inversely proportional to the angular momentum. Since $L = I\omega$, we have

$\omega_p \propto 1/\omega$, which means that as the gyroscope gradually spins slower around its axis (due to friction forces), it will precess faster around the vertical. You may have observed this behavior in a gyroscope or a spinning top.

54. **INTERPRET** We use conservation of angular momentum to find the radius of a white dwarf star. We know the initial radius, mass, and rotational speed; so this gives us the initial angular momentum. The final angular momentum will be the same, so we use it to find the radius knowing the final mass and angular speed.

DEVELOP We're told that the star collapses with 60% of its original mass. That means 40% of the mass is "blown off." We'll assume these outer layers take their angular momentum with it. So we'll only deal with conservation of momentum in the star's core with $M = 0.6\,M_{\text{sun}}$. Assuming the star is uniform, this core initially occupies a sphere with radius:

$$R_0 = \sqrt[3]{\frac{M}{\frac{4\pi}{3}\rho_{\text{sun}}}} = \sqrt[3]{\frac{0.6M_{\text{sun}}}{\frac{4\pi}{3}\left(M_{\text{sun}}/\frac{4\pi}{3}R_{\text{sun}}\right)}} = \sqrt[3]{0.6}R_{\text{sun}}$$

EVALUATE Before the collapse, the core's angular momentum is given by $L = I_0\omega_0$, where $I_0 = \frac{2}{5}MR_0^2$, and $\omega_0 = 2\pi/25$ d. After the collapse, the core still has the same angular momentum, but the expression is now $L = I\omega$, where $I = \frac{2}{5}MR^2$, and $\omega = 2\pi/131$ s. Solving for the unknown final radius, we get:

$$R = R_0\sqrt{\frac{\omega_0}{\omega}} = \left(\sqrt[3]{0.6}R_{\text{sun}}\right)\sqrt{\frac{131\text{ s}}{25\cdot 24\cdot 3600\text{ s}}} = 6.57\times 10^{-3}R_{\text{sun}} = 4.57\times 10^6\text{ m}$$

This radius is about 70% of the radius of the Earth, and 150 times smaller than the original star.

ASSESS One could assume that the outer layers blow off without taking away any of the angular momentum, and the core inherits *all* of the original angular momentum of the star before the collapse: $L = \frac{2}{5}M_{\text{sun}}R_{\text{sun}}^2\omega_0$. (This is unlikely but it can serve as an upper bound.) In such a case, the final radius would be 100 times smaller than the original star.

55. **INTERPRET** We are asked to determine what happens to a spinning gyroscope when different torques are applied to it.

DEVELOP Initially, the gyroscope has no torque on it, and the angular velocity and angular momentum both point to the right. By applying a force on the gyroscope between the arrowhead and disk, you exert a torque given by $\vec{\tau} = \vec{r}\times\vec{F}$ (Equation 11.2).

EVALUATE In this case, the force $\vec{F}$ points into the page and is applied at a radius $\vec{r}$ that points to the right. By the right-hand rule, the torque points upward. By Equation 11.5 $\left(d\vec{L}/dt = \vec{\tau}\right)$, the angular momentum will move in the torque's direction. Because the arrowhead points in the direction of the angular momentum, it too will move upward.

The answer is (d).

ASSESS It might seem odd that you push something in one direction, and it moves in a perpendicular direction. But this is just how the rotational analog of Newton's second law works.

56. **INTERPRET** We are asked to determine what happens to a spinning gyroscope when different torques are applied to it.

DEVELOP As described in the previous problem, the push results is a torque $\vec{\tau} = \vec{r}\times\vec{F}$.

EVALUATE In this case, the force $\vec{F}$ points upward and is applied at a radius $\vec{r}$ that points to the right. By the right-hand rule, the torque points out of the page. The angular momentum vector and the arrowhead will both move toward you, out of the page.

The answer is (b).

ASSESS Compared to the previous problem, the force has rotated by 90°, so we'd expect the torque would as well.

57. **INTERPRET** We are asked to determine what happens to a spinning gyroscope when different torques are applied to it.

DEVELOP The added weight means the gyroscope is no longer balanced on the stand. There will be more downward force on the left-side than on the right-side. By the right-hand rule, this generates a torque that points out of the page.

EVALUATE This torque will cause the angular momentum to move slightly in the direction of the torque, i.e. out of the page. Recall Figure 11.9, where $\Delta\vec{L}$ points the same way as $\vec{\tau}$. This shift in the angular momentum will start the gyroscope turning in a clockwise direction as seen from above. As it moves, the torque will change so that the gyroscope continues to precess clockwise about the stand.

The answer is (d).

ASSESS One might have wrongly assumed that since the torque is out of the page it will "push" the left-hand side of the gyroscope (where the weight was added), thus resulting in a counter-clockwise rotation. The torque does not act on a specific point, but instead acts on the whole system through its angular momentum.

58. **INTERPRET** We are asked to determine what happens to a spinning gyroscope when different torques are applied to it.

DEVELOP As the gyroscope precesses over a short time interval, Δt, the angular momentum changes by a small amount: $\Delta\vec{L} = \vec{\tau}\cdot\Delta t$. This shift corresponds to a change in the direction of $\vec{L}$ characterized by an angle: $\Delta\phi = \Delta L / L$ (see Problem 11.53 for a similar case).

EVALUATE The precession rate is equal to the rate at which $\Delta\phi$ changes with time:

$$\omega_\text{p} = \frac{\Delta\phi}{\Delta t} = \frac{\Delta L}{\Delta t \cdot L} = \frac{\tau}{L} = \frac{\tau}{I\omega}$$

This shows that the precession rate is inversely proportional to the rotation rate of the disk, ω. So if the rotation rate increases, the precession rate will decrease.

The answer is (a).

ASSESS We can check the units on our expression for the precession rate. The ratio τ / I is equal to the angular acceleration, α (recall Equation 10.11). So the units are

$$\left[\omega_\text{p}\right] = \frac{[\alpha]}{[\omega]} = \frac{1/\text{s}^2}{1/\text{s}} = 1/\text{s}$$

This is what we would expect for the precession rate.