The production Function

Firms can turn inputs into outputs in a variety of ways, using various combinations of labor, materials, and capital. We can describe the relationship between the input into the production process and the resulting output by the production function.

Production function: The mathematical relationship between inputs and outputs.

A *production function* indicates the highest output *q* that a firm can produce for every specified combination of inputs. Although in practice firms use a wide variety of inputs, we will keep our analysis simple by focusing on only two, labor *L* and capital *K*. we can then write the production function as:

$$q = F(K, L)$$

This equation relates the quantity of output to the quantities of the two inputs, capital and labor. Because the production function allows input to be combined in varying proportions, output can be produced in many ways. For the production function equation this could mean using more capital and less labor, or vice versa.

Average and Marginal Products

Average product: Output per unit of a particular input.

The average product is calculated by dividing the total output q by the input of labor L. *The average product of labor measures the productivity* of the firm workforce in terms of how much output each worker produce in average.

Average product of labor $(APL) = \frac{Output}{labor\ input} = \frac{q}{L}$

Marginal product: Additional output produced as an input is increased by one unit.

Marginal product of labor $(MPL) = \frac{Change\ in\ Output}{Change\ in\ labor\ input} = \frac{\Delta q}{\Delta L} = \frac{\partial q}{\partial L}$

Marginal product of capital (MPK) is the extra output obtained by using one more machine while holding the number of workers constant.

Marginal product of labor $(MPK) = \frac{Change\ in\ Output}{Change\ in\ capital\ input} = \frac{\Delta q}{\Delta K} = \frac{\partial q}{\partial K}$

Example

Suppose that the hourly output of chili at a barbecue (q measured in pounds) is characterized by: $q = 2KL + L$. where K is the capital input used each hours, and L is the number of worker hours employed. Suppose also that the amount of capital is fixed at 12.

a. How much L is needed to produce 100 pound per hour?

$q = L(2K + 1)$ ⇨ $100 = L(2 * 12 + 1)$ ⇨ $100 = 25L$ ⇨ $L = 100/25 = 4$

b. What is the average product of labor to produce 100 pound per hour?

When K = 12, and q = 100 ⇨ L = 4

Average product of labor (APL) = $\frac{Output}{labor\ input} = \frac{q}{L} = \frac{100}{4} = 25$

c. What is the marginal product of labor when the firm employing 100 workers?

Marginal product of labor (MPL) = $\frac{\partial q}{\partial L} = 2K + 1 = 2(12) + 1 = 25$

<u>Example</u>
A firm produces output according to a production function: $q(K,L) = min\{4K, 8L\}$.

1. How much output is produced when K = 2, L = 3?

$q(K,L) = min\{4K, 8L\} \quad \rightarrow \quad q = \min\{4*2, 8*3) = \min\{8,24\} = 8$

2. What is the average product of labor when K = 3, L = 3?

$q(K,L) = min\{4K, 8L\} \quad \rightarrow \quad q = \min\{4*3, 8*3) = \min\{12,24\} = 12$

$APL = \frac{q}{L} = \frac{12}{3} = 4\ units$

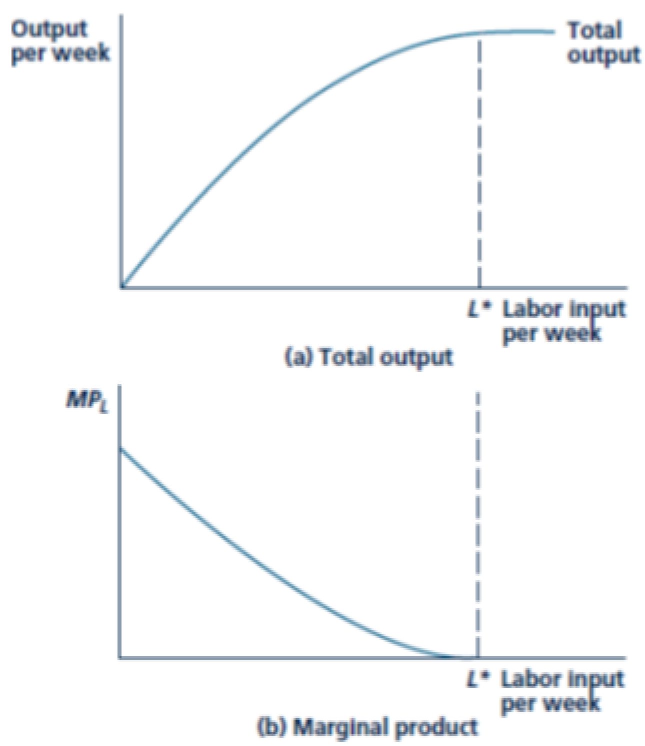
<u>Example</u>
A firm produces an output using production function: $q = \sqrt{LK}$. What is the marginal product of labor (MPL) when the firm employs 3 labors and 3 capitals?

Marginal product of labor (MPL) = $\frac{\partial q}{\partial L} = \frac{K}{2\sqrt{LK}} = \frac{3}{2\sqrt{3*3}} = \frac{3}{2*3} = \frac{3}{6} = \frac{1}{2}$

<u>Diminishing Marginal Product</u>

Adding new workers increases output significantly, but these gains diminish as even more labor is added and the fixed amount of capital becomes over utilized. The concave shape of the total output curve in panel a therefore reflects the economic principle of diminishing marginal product.

<u>Marginal Product Curve</u>
A geometric interpretation of the marginal product concept is straightforward—it is the slope of the total product curve. The decreasing slope of the curve shows diminishing marginal product. For higher values of labor input, the total curve is nearly flat—adding more labor raises output only slightly.



Panel a shows the relationship between output and labor input, holding other inputs constant. Panel b shows the marginal product of labor input, which is also the slope of the curve in panel a. Here, MPL diminishes as labor input increases. MPL reaches zero at L*.

Example

A firm produces an output using production function: $q = L^2 K$.

Calculate the marginal product of labor. Does the marginal product of labor diminish, remain constant, or increase as the firm employs more workers? Explain.

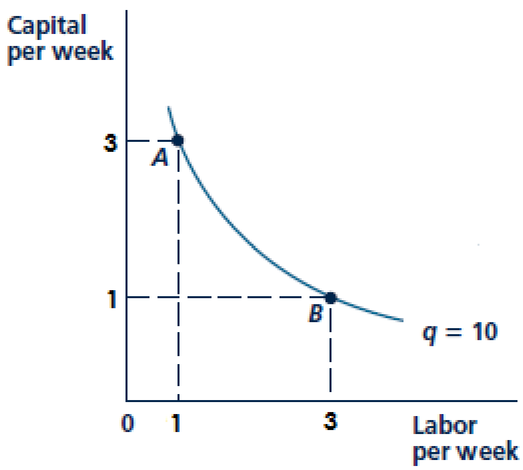
$$MPL = \frac{\partial q}{\partial L} = 2KL$$

As the firm employs more workers ($L \uparrow$), amount of capital decrease ($K \downarrow$)
So as L increase and K decrease the MPL may be diminishes

Isoquant

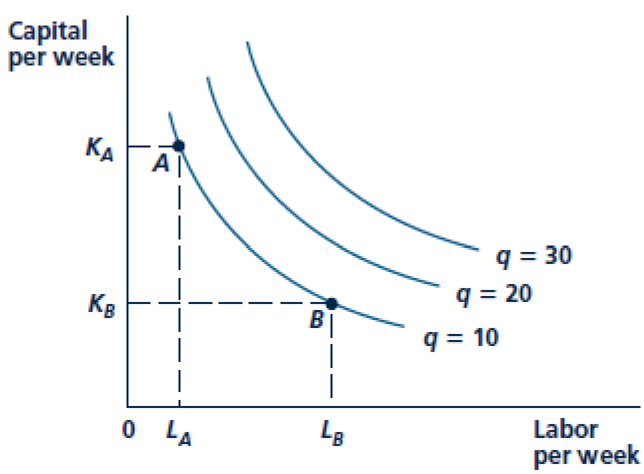
A curve that shows the various combinations of inputs that will produce the same amount of output.

For example, isoquant q1 shows all combinations of labor and capital per year those together yield 10 units of output per year. Two of these points, A and B. At A, 1 unit of labor and 3 units of capital yield 10 units of output; at B, the same output is produced from 3 units of labor and 1 unit of capital.



Isoquant map

Graph combining a number of isoquants, used to describe a production function.



Output increases as we move from isoquant q1 (at which 10 units are produced), to isoquant q2 (20 units), and to isoquant q3 (30 units).

Example:

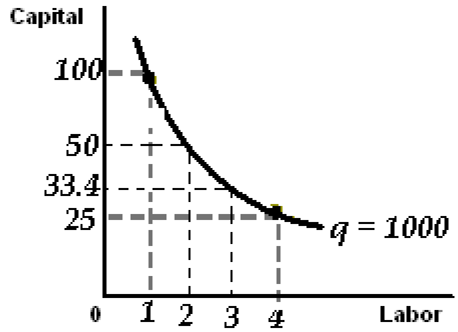
The production function for puffed rice is given by: $q = 100\sqrt{L\,K}$

Where q is the number of boxes produce per hour, K is the number of puffing guns used each hour, and L is the number of workers hired each hour.

a. Calculate the q = 1,000 isoquant for this production function and show it on a graph.

$$q = 100\sqrt{L\,K} \;\Rightarrow\; 1{,}000 = 100\sqrt{L\,K} \;\Rightarrow\; 10 = \sqrt{L\,K} \;\Rightarrow\; 100 = LK \;\Rightarrow\; K = \frac{100}{L}$$

L	K
1	100
2	50
3	33.4
4	25



Marginal rate of technical substitution (RTS)

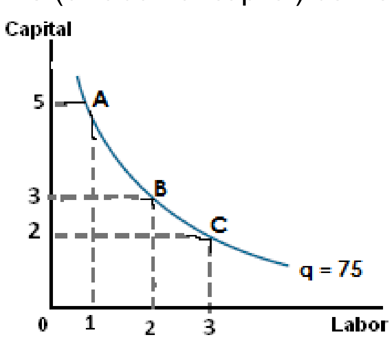
The amount by which one input can be reduced when one more unit of another input is added while holding output constant. The negative of the slope of an isoquant.

The slope of each isoquant indicates how the quantity of one input can be traded off against the quantity of the other, while output is held constant. When the negative sign is removed, we call the slope the *marginal rate of technical substitution (RTS).*

RTS = $\dfrac{-\,Change\ in\ capital\ input}{Change\ in\ labor\ input} = \dfrac{-\,\Delta K}{\Delta L}$ for a fixed level of q

The isoquant has a negative slope) because the firm can decrease its use of capital if one more unit of labor is employed.

The RTS (of labor for capital) between points A and B:



RTS = $\dfrac{-\Delta \mathrm{K}}{\Delta \mathrm{L}} = \dfrac{-(3-5)}{(2-1)} = \dfrac{2}{1} = 2$

This means that if the firm adds one unit of labor, the firm should decrease the used of capital by 2 units to keep output level constant.

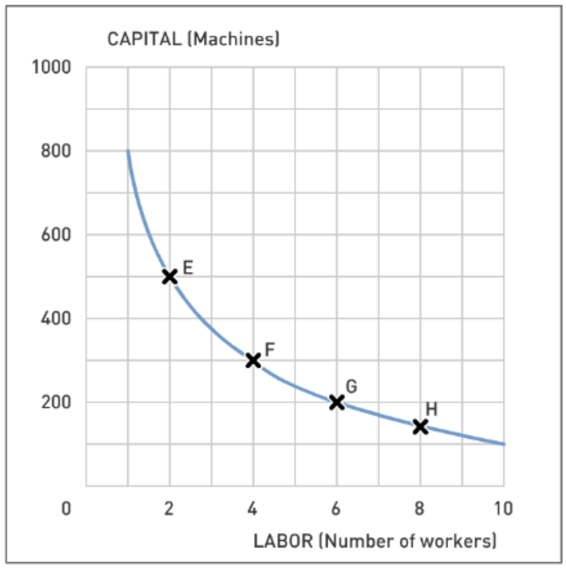
Diminishing RTS:

Along any isoquant the (negative) slope become flatter and the RTS diminishes. The MRTS falls as we move down along an isoquant. The mathematical implication is that isoquants are convex or bowed inward.

The diminishing MRTS tells us that the productivity of any one input is limited. As more and more labor is added to the production process in place of capital, the productivity of labor falls.

Example:

The following graph shows the isoquant representing the combinations of capital and labor needed to produce 100 units.



Does the RTS diminish, remain constant, or increase as the firm employs more workers? Explain.

RTS (between point E and F) = $\frac{-\Delta K}{\Delta L} = \frac{-(300-500)}{(4-2)} = \frac{200}{2}$ = 100

RTS (between point F and G) = $\frac{-\Delta K}{\Delta L} = \frac{-(200-300)}{(6-4)} = \frac{100}{2}$ = 50

RTS (between point G and H) = $\frac{-\Delta K}{\Delta L} = \frac{-(150-200)}{(8-6)} = \frac{50}{2}$ = 25

As number of labor increase, RTS decrease → RTS diminishing

The RTS and Marginal Products

The RTS is closely related to the marginal products of labor MPL and capital MPK. To see how, imagine adding some labor and reducing the amount of capital sufficient to keep output constant. The additional output resulting from the increased labor input is equal to the additional output per unit of additional labor (the marginal product of labor) times the number of units of additional labor:

Additional output from increased use of labor = $(MP_L)(\Delta L)$

Similarly, the decrease in output resulting from the reduction in capital is the loss of output per unit reduction in capital (the marginal product of capital) times the number of units of capital reduction:

Reduction in output from decreased use of capital = $(MP_K)(\Delta K)$

Because we are keeping output constant by moving along an isoquant, the total change in output must be zero. Thus,

$(MP_L)(\Delta L)\ (MP_K)(\Delta K) = 0$

Now, by rearranging terms we see that

$$\frac{\text{MPL}}{\text{MPK}} = \frac{-\Delta K}{\Delta L} = RTS$$

$$RTS = \frac{\text{MPL}}{\text{MPK}}$$

This equation tells us that the marginal rate of technical substitution between two inputs is equal to the ratio of the marginal products of the inputs.

Example

A firm's marginal product of labor is 4 and its marginal product of capital is 5. if the firm adds one unit of labor, but does not want its output quantity to change, the firm should:
(a) Use five fewer units of capital.
(b) Use 0.8 fewer units of capital
(c) Use 1.25 fewer units of capital
(d) Add 1.25 units of capital

MRTS (of L for K) = $\frac{\text{MPL}}{\text{MPK}} = \frac{4}{5} = 0.8$

MRTS = 0.8 means that if the firm adds one unit of labor, then the firm should decrease the used of capital by 0.8 unit to keep output level constant.

Example

The production function for soy beans is q =10 K L. calculate the RTS of labor for capital when the firm using 4 labor and 2 capital.

RTS (of L for K) = $\frac{\text{MPL}}{\text{MPK}}$

MPL = $\frac{\partial \text{q}}{\partial \text{L}}$ = 10K

MPK = $\frac{\partial \text{q}}{\partial \text{K}}$ = 10L

RTS (of L for K) = $\frac{\text{MPL}}{\text{MPK}} = \frac{10K}{10L} = \frac{K}{L} = \frac{2}{4}$ = ½

Example

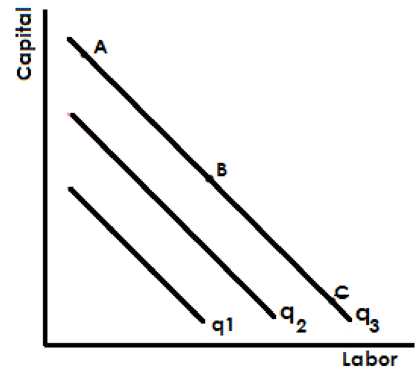
The production function for a good is given by $q = 4\,K\,L^2$. calculate the RTS of labor for capital when the firm using 10 labor and 2 capital.

RTS = $\frac{\text{MPL}}{\text{MPK}} = \frac{8\,KL}{4\,L^2} = \frac{2K}{L} = \frac{2(2)}{10} = \frac{4}{10} = 0.4$

Production Functions–Two Special Cases

Input substitution (labor and capital are substitute)

Another important characteristic of a production function is how ‘‘easily’’ capital can be substituted for labor, or, more generally, how any one input can be substituted for another. This characteristic depends primarily on the shape of a single isoquant. So far we have assumed that a given output level can be produced with a variety of different input mixes–that is, we assumed firms could substitute labor for capital while keeping output constant.



The production functions for these goods: $q(L,K) = a\,L + b\,K$

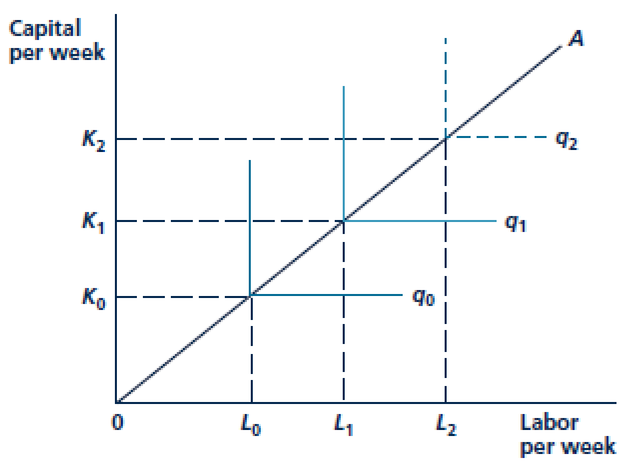
When the isoquants are straight lines, the RTS is constant. Thus the rate at which capital and labor can be substituted for each other is the same no matter what level of inputs is being used.

Fixed-Proportions Production Function ((labor and capital are complement)

Fixed-proportions production function: A production function in which the inputs must be used in a fixed ratio to one another.

It may be the case that absolutely no substitution between inputs is possible. This case is shown in figure. If K_1 units of capital are used, exactly L_1 units of labor are required to produce q_1 units of output. If K_1 units of capital are used and less than L_1 units of labor are used, q_1 cannot be produced

If K_1 units of capital are used and more than L_1 units of labor are used, no more than q_1 units of output are produced. With $K = K_1$, the marginal physical product of labor is zero beyond L_1 units of labor. The q_1 isoquant is horizontal beyond L_1. Similarly, with L_1 units of labor, the marginal physical product of capital is zero beyond K_1 resulting in the vertical portion of the isoquant. This type of production function is called a *fixed-proportion production function* because the inputs must be used in a fixed ratio to one another.



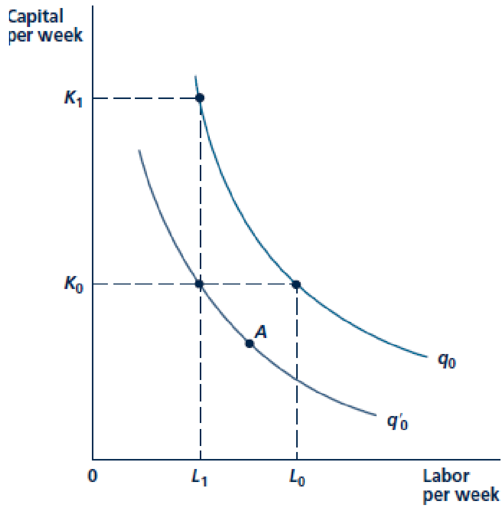
The mathematical notation is representing by: $q(K,L) = min(K,L)$

Changes in Technology

Technical progress is a shift in the production function that allows a given output level to be produced using fewer inputs.

The improvement in technology is represented in the Figure by the shift of the q_0 isoquant to q'_0.

Technical progress shifts the q_0 isoquant to q'_0. Whereas previously it required K_0, L_0 to produce q_0 now, with the same amount of capital, only L_1 units of labor are required. This result can be contrasted to capital-labor substitution, in which the required labor input for q_0 also declines to L_1 and more capital (K_1) is used.



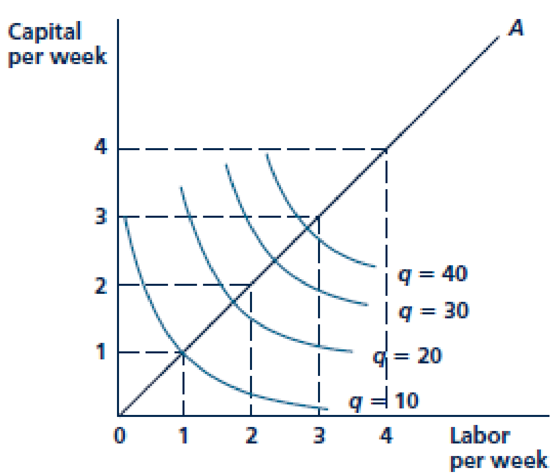
Returns to Scale

Returns to scale is the rate at which output increases in response to proportional increases in all inputs.

Increasing Return to Scale عوائد الحجم المتزايد

If output more than doubles when inputs are doubled, there are increasing returns to scale. This might arise because the larger scale of operation allows managers and workers to specialize in their tasks and to make use of more sophisticated, large-scale factories and equipment. The automobile assembly line is a famous example of increasing returns.

عوائد الحجم المتزايد تشير الى ان زيادة جميع عناصر الانتاج بنسبة 10% مثلاً تؤدي الى زيادة الكمية المنتجة بنسبة اكبر من 10% (نسبة التغيير في كمية الانتاج اكبر من نسبة التغيير في مدخلات الانتاج)



Increasing returns to scale

Constant Return to Scale عوائد الحجم الثابت

Situation in which output doubles when all inputs are doubled are called constant return to scale. With constant returns to scale, the size of the firm's operation does not affect the productivity of its factors: Because one plant using a particular production process can easily be replicated, two plants produce twice as much output. For example, a large travel agency might provide the same service per client and use the same ratio of capital (office space) and labor (travel agents) as a small agency that services fewer clients.

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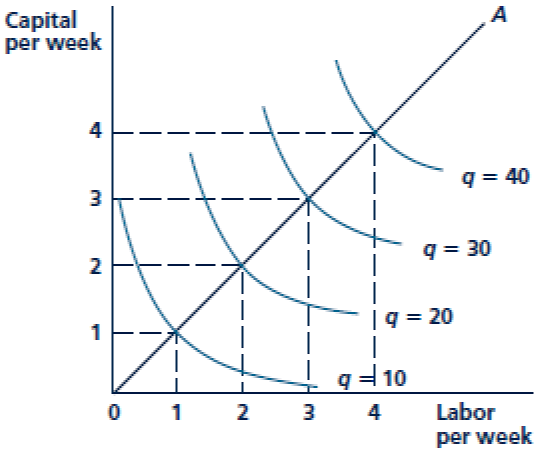

Constant returns to scale

Decreasing returns to scale عوائد الحجم المتناقص

Situation in which output less than doubles when all inputs are doubled are called constant return to scale. This case of decreasing returns to scale applies to some firms with large-scale operations. Eventually, difficulties in organizing and run- rung a large-scale operation may lead to decreased productivity of both labor and capital. Communication between workers and managers can become difficult to monitor as the workplace becomes more impersonal. Thus, the decreasing-returns case is likely to be associated with the problems of coordinating tasks and maintaining a useful line of communication between management and workers.

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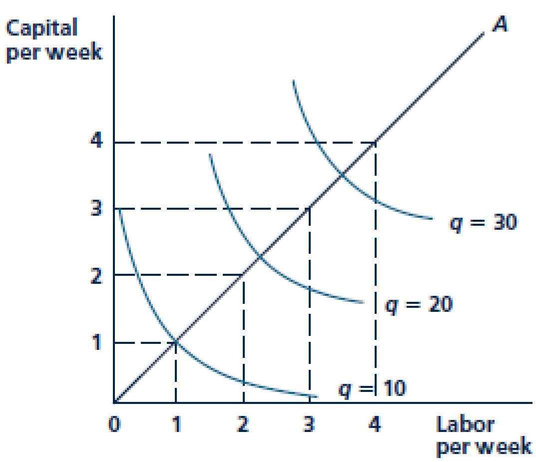

Decreasing returns to scale

Numerical Examples

1. The production function for soy beans is $q = 40\sqrt{LK}$. Dose this production function have constant, increasing or decreasing return to scale?

   q (K,L) = $40\sqrt{LK}$, when we doubling of all inputs ⇨ q(2K,2L) = 40 $\sqrt{(2L)(2K)}$

   ⇨ q(2K,2L) = 40 $\sqrt{4LK}$ = 2 x $40\sqrt{LK}$ = 2q ⇨ constant return to scale.

2. The production function for soy beans is q = KL. Dose this production function have constant, increasing or decreasing return to scale?

   q (K,L) = K L, when we doubling of all inputs ⇨ q(2K,2L) = (2K) x (2L)

   ⇨ q(2K,2L) = (2K) x (2L)= 4 (KL) = 4q >2q ⇨ Increasing return to scale.

3. The production function for soy beans is q = 2K + L. Dose this production function have constant, increasing or decreasing return to scale?

   q (K,L) = 2K + L, when we doubling of all inputs ⇨ q(2K,2L) = 2(2K) + (2L)

   q(2K,2L) = 2 (2K + L) = 2q ⇨ constant return to scale.

4. The production function for soy beans is q = $K^{0.5} L^{0.3}$. Dose this production function have constant, increasing or decreasing return to scale?

   q (K,L) = $K^{0.5} L^{0.3}$, when we doubling of all inputs ⇨ q(2K,2L) = $(2K)^{0.5} (2L)^{0.3}$

   ⇨ q(2K,2L) = $(2)^{0.5} (K)^{0.5} (2)^{0.3} (L)^{0.3}$ = $2^{0.8}\{(K)^{0.5} (L)^{0.3}\}$= $2^{0.8}$ q < 2q ⇨ decreasing return to scale.

If the productions function is given by: q = $K^{\alpha} L^{\beta}$ where $0 \leq \alpha, \beta \leq 1$, is called a Cobb-Douglas production function.

- If α + β = 1 ⇨ the productions function constant return to scale
- If α + β > 1 ⇨ the productions function increasing return to scale
- If α + β < 1 ⇨ the productions function decreasing return to scale

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Questions:

1. The production function for the personal computers of DISK, Inc., is given by: q = $10K^{0.5} L^{0.5}$. Where q is the number of computers produced per day, K is hour of machine time, and L is hours of labor input. DISK's competitor, FOLPPY, Inc., is using the production function: q = $10K^{0.6} L^{0.4}$.
   a. If both companies use the same amounts of capital and labor, which will generate more output?
   b. Assume that capital is limited to 9 machine hours, but labor is unlimited in supply. In which company is the marginal product of labor greater? Explain.

2. Do the following function exhibit increasing, constant, or decreasing return to scale?
   a. q = 3L + 2K
   b. q = $3LK^2$
   c. q = $(2L + 2K)^{0.5}$
   d. q = $4L^{0.5} + 4K$

3. The product function is q = $2\sqrt{LK}$.
   a. Calculate the q = 8 isoquant for this production function and show it on a graph.
   b. Find the marginal rate of technical substitution (MRTS) on that isoquant curve q=8 when L=1.

4. A firm has a production function given by: q = LK + 2L. Suppose that the amount of capital is fixed at K = 4.
   a. Is the marginal product of labor diminishing? Why?
   b. Find the returns to scale of this production function
   c. Find the MRTS when K = 3, and L =10.