10.1 Sequences

Representing sequences

A sequence is a List of numbers

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

in a given Order

Each $a_1, a_2, \ldots, a_n, \ldots$ represents a number and these are the terms of the sequence.

Example:

$2, 4, 6, 8, 10, \ldots, 2n, \ldots$

$a_1 = 2$

$a_2 = 4$

$a_3 = 6$

$\vdots$

$a_n = 2n$

n is called the index of a_n

Order is important

$2, 4, 6, 8, \ldots$ is not the same as $4, 2, 6, 8, \ldots$

We can think of the sequence $a_1, a_2, \ldots, a_n, \ldots$ as a function sends the 1 to a_1

2 to a_2

$\vdots$

n to a_n

Examples:

1. $12, 14, 16, 18, 20, 22, \dots$

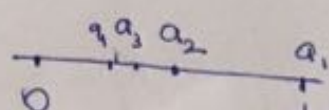
   $a_n = 10 + 2n, \quad n = 1, 2, 3, \dots$

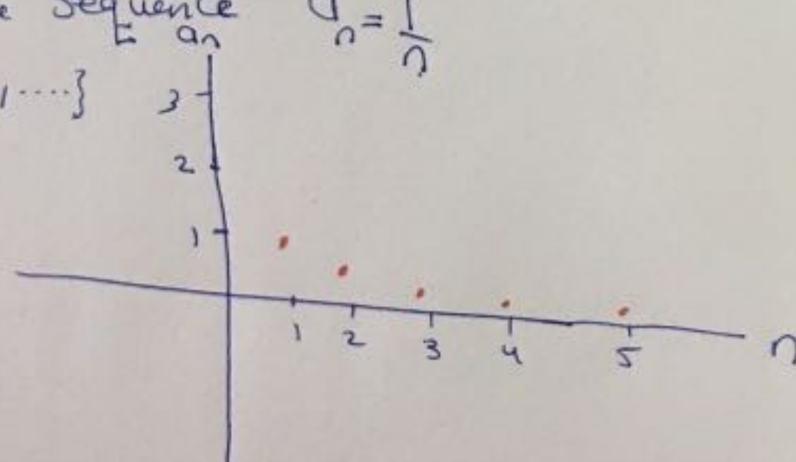
2. $a_n = \sqrt{n} = \{1, \sqrt{2}, \sqrt{3}, 2, \dots, \sqrt{n}, \dots\}$
3. $b_n = (-1)^{n+1} \cdot \frac{1}{n} = \{1, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{4}, \frac{1}{5}, \dots, (-1)^{n+1}\frac{1}{n}, \dots\}$
4. $d_n = (-1)^{n+1} = \{1, -1, 1, -1, 1, -1, \dots\}$

also we can write a_n as $\{\sqrt{n}\}_{n=1}^{\infty}$

Example: Represent the sequence $a_n = \frac{1}{n}$

$a_n = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\}$





Convergence and Divergence.

$a_n = \frac{1}{n} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots, \frac{1}{n}, \dots \right\}$

$$a_n \longrightarrow 0$$

$b_n = \sqrt{n} = \left\{ 1, \sqrt{2}, \sqrt{3}, \sqrt{4}, \dots, \sqrt{n}, \dots \right\}$

$c_n = (-1)^{n+1} = \left\{ 1, -1, +1, -1, \dots \right\}$ "alternating"

sequence

converge   diverge.

Definition.

The sequence $\{a_n\}$ **converges** to the number L if for every positive number ϵ, there corresponds an integer N such that for all n

$$n > N \rightarrow |a_n - L| < \epsilon$$

If no such L exists we say that $\{a_n\}$ **diverges**

Example:

① $\{a_n\} = \{\frac{1}{n}\}$ Converge to 0

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

② $b_n = k$ where k is a constant

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} k = k$$ so $\{b_n\}$ converge.

③ $c_n = (-1)^{n+1}$

$c_n = \{1, -1, 1, -1, 1, -1, \dots\}$ "alternating"

$\lim_{n\to\infty} c_n$ Doesn't exist so $\{c_n\}$ diverges

Calculating Limit of sequences

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real number.
and let A and B be real numbers

If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$ then.

1. $\lim_{n\to\infty} (a_n + b_n) = A + B$
2. $\lim_{n\to\infty} a_n - b_n = A - B$

3. $\lim_{n\to\infty} k.b_n = k.B$ "k is any number"

4. $\lim_{n\to\infty} a_n . b_n = A.B$

5. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$. $B \neq 0$

Example:

$$\lim_{n\to\infty} \frac{-1}{n} = -1 . \lim \frac{1}{n} = -1.0 = 0$$

$$\lim_{n\to\infty} \left(\frac{n-1}{n}\right) = \lim_{n\to\infty}\left(1-\frac{1}{n}\right) = \lim_{n\to\infty} 1 - \lim_{n\to\infty}\frac{1}{n} = 1-0 = 1$$

$$\lim_{n\to\infty} \frac{5}{n^2} = 5 \lim_{n\to\infty}\frac{1}{n} . \lim_{n\to\infty}\frac{1}{n} = 5.0.0 = 0$$

$$\lim_{n\to\infty} \frac{4-7n^6}{n^6+3} = -7$$

Remarks:

① If $\{a_n\} = \{1,2,3,4,\dots\}$ "Diverges

$a_n = n$

$\lim_{n\to\infty} a_n = \infty$

$\{b_n\} = \{-1,-2,-3,-4,\dots\}$ Diverges

$b_n = -n \Rightarrow \lim b_n = -\infty$

$a_n + b_n$ converge to 0

$\{a_n + b_n\} = \{0, 0, 0, \ldots, 0\}$

(2) Any constant "nonzero" multiple of a divergent series $\{a_n\}$ is divergent

Theorem 2: The Sandwich Theorem for sequences

Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers. If $a_n \le b_n \le c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then

$$\lim_{n\to\infty} b_n = L$$

Example:

a) $\frac{\cos n}{n} \to 0$

$-1 \le \cos n \le 1$

$$-\frac{1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}$$

$\downarrow \qquad \Downarrow \qquad \downarrow$

$0 \qquad 0 \qquad 0$

b) $\frac{1}{2n} \to 0$

$\frac{1}{2n} = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$

~~$\frac{1}{2n}$~~ $\frac{1}{n} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$

so $0 < \frac{1}{2n} < \frac{1}{n}$

$0 \to 0$, $\frac{1}{2n} \Rightarrow 0$, $\frac{1}{n} \to 0$

c) $(-1)^n \cdot \frac{1}{n} \to 0$

$-\frac{1}{n} \leq (-1)^n \cdot \frac{1}{n} \leq \frac{1}{n}$

$-\frac{1}{n} \to 0$, $(-1)^n \cdot \frac{1}{n} \Rightarrow 0$, $\frac{1}{n} \to 0$

Theorem 8: The Continuous Function Theorem for sequence

Let $\{a_n\}$ be a sequence of real numbers

If $a_n \to L$

and if f is a continuous function at L and defined at all a_n

then $f(a_n) \to f(L)$

Example: Show that $\sqrt{\frac{n+1}{n}} \to 1$

$$\frac{n+1}{n} \to 1$$

$$\sqrt{\frac{n+1}{n}} \to \sqrt{1} = 1$$

Example: $2^{\frac{1}{n}} \to ??$

take $a_n = \frac{1}{n}$

$a_n \to 0$

$f(x) = 2^x$

$f(a_n) = 2^{a_n} = 2^{\frac{1}{n}}$

$f(a_n) \to 2^0 = 1$