The nth term Test for a Divergent series

(1)

Example: The series $\sum_{n=1}^{\infty} \frac{n+1}{n} = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \frac{5}{4} + \cdots + \frac{n+1}{n} + \cdots$

diverges since the partial sums eventually out grow every preassigned number.
Each term is greater than 1, so the sum of n terms is greater than n.

If the series $\sum_{n=1}^{\infty} a_n$ converge, then $\lim_{n\to\infty} a_n = 0$.

why? $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$

$$S_n = a_1 + a_2 + \cdots + a_n$$

$$S_{n-1} = a_1 + a_2 + \cdots + a_{n-1}$$

If $\sum_{n=1}^{\infty} a_n$ converges, this means $S = \text{sum} = L$ Fixed #

so S_n and S_{n-1} are very close to $S = \text{sum}$ as $n \to \infty$

so $S_n - S_{n-1} \longrightarrow 0$ [their difference is very close to zero]

$$\underbrace{S_n - S_{n-1}}_{\downarrow \atop 0} = \underbrace{a_n}_{\Downarrow \atop 0}$$

So If the series $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$

$\longrightarrow$ Th 7

Nth Term Test for Divergence.

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero (S)

Example:

(1) $\sum_{n=1}^{\infty} n^2$ diverges

$\lim_{n\to\infty} n^2 = \infty$, by nth-term test the series diverges

(2) $\sum_{n=1}^{\infty} \frac{n+1}{n}$

$\lim_{n\to\infty} \frac{n+1}{n} = 1$

So $\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges [nth term test]

(3) $\sum_{n=1}^{\infty} (-1)^{n+1}$

$\lim_{n\to\infty} (-1)^{n+1}$ DNE

So $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges

Example:

$$1+\underbrace{\frac{1}{2}+\frac{1}{2}}_{2}+\underbrace{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}_{4}+\underbrace{\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}}_{5}+\cdots+\underbrace{\frac{1}{2^n}+\cdots+\frac{1}{2^n}}_{2^n\, n}+\cdots$$

$$= 1 \qquad = 1 \qquad = 1$$

Diverge

Th 8: If $\sum a_n = A$ and $\sum b_n = B$ are convergent, then

(1) Sum Rule: $\sum a_n + b_n = \sum a_n + \sum b_n = A + B$

(2) Difference Rule $\sum a_n - b_n = \sum a_n - \sum b_n = A - B$

(3) Constant Multiple Rule: $\sum k a_n = k \sum a_n = kA$

Proof (1)

$A_n = a_1 + a_2 + a_3 + \cdots + a_n$ nth partial sum of $\sum a_n$

$B_n = b_1 + b_2 + \cdots + b_n$ nth partial sum of $\sum b_n$

$S_n =$ nth partial sum of $\sum (a_n + b_n)$

$$S_n = (a_1+b_1) + (a_2+b_2) + \cdots + (a_n+b_n)$$

$$= (a_1 + a_2 + \cdots + a_n) + (b_1 + b_2 + \cdots + b_n)$$

$$= A_n + B_n$$

$\sum a_n = A$ so $A_n \to A$

$\sum b_n = B$ so $B_n \to B$

So $\boxed{S_n \to A + B}$

Corollary:

① Every nonzero constant multiple of a divergent series diverges

② If $\sum a_n$ converges and $\sum b_n$ diverges then $\sum(a_n + b_n)$ and $\sum(a_n - b_n)$ both diverge.

Example:

$$\sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{3}{6}\right)^{n-1} - \frac{1}{6^{n-1}}$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{6}\right)^{n-1}$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} - \sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^{n-1}$$

$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$ geometric
$a = 1$
$r = \frac{1}{2}$
conv.

, $\sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^{n-1}$ geometric
$a = 1$
$r = \frac{1}{6}$
conv.

So $\sum = \frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{1}{6}} = 2 - \frac{6}{5}$

$= \frac{4}{5}$

37 $\sum_{n=1}^{\infty} \ln(\sqrt{n+1}) - \ln\sqrt{n}$

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$S_n = \ln\sqrt{2} - \ln 1 + \ln\sqrt{3} - \ln\sqrt{2} + \cdots + \ln\sqrt{n} - \ln\sqrt{n-1} + \ln\sqrt{n+1} - \ln\sqrt{n}$

$S_n = \ln\sqrt{n+1}$

$S_n \longrightarrow \infty$ as $n \to \infty$

so the series diverges

51 $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{3}{2^n}$

$= \frac{3}{2} - \frac{3}{2^2} + \frac{3}{2^3} \mp \cdots$

$a = \frac{3}{2}$

$r = \frac{-1}{2}$

$\text{sum} = \frac{\frac{3}{2}}{1 - \frac{-1}{2}} = \frac{\frac{3}{2}}{\frac{3}{2}} = 1$