10.2 Infinite series

An infinite series is the infinite sum of numbers

$$a_1 + a_2 + a_3 + \dots + a_n + \dots$$

nth partial sum

$$S_n = a_1 + a_2 + a_3 + \dots + a_n$$

As n gets larger we expect the sums to get closer and closer to a limiting value. in the same sense that the terms of a sequence approach a limit

Example:

we have the series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$$

$S_1 = 1 \quad = 1$ (1st Partial sum)

$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$ (2nd partial sum)

$S_3 = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$ (3rd partial sum)

$\vdots \qquad\qquad\qquad \vdots$

$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{n-1}}$ (nth partial sum)

Definition:

Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an infinite series

The number a_n is the nth term of the series.

The sequence $\{s_n\}$ is defined by

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$
$$\vdots$$
$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

is the sequence of partial sums of the series

The number s_n = nth partial sum

If the sequence of partial sums converges to a limit L, we say that the series converges and that its sum is L.

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L$$

If the sequence of partial sums of the series doesn't converge, then the series diverges

Geometric Series

geometric series are of the form

$$a + ar + ar^2 + ar^3 + \dots + ar^{n-1} + \dots = \sum_{n=1}^{\infty} ar^{n-1}$$

a and r are fixed real numbers and $a \neq 0$

we can write it as $\sum_{n=0}^{\infty} ar^n$

Example:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots + \left(\frac{1}{2}\right)^{n-1} + \dots$$

$a = 1$

$r = \frac{1}{2}$ "positive ratio"

Example:

$$1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \dots + \left(\frac{-1}{3}\right)^{n-1} + \dots$$

$a = 1$

$r = \frac{-1}{3}$ "r = ratio is negative"

If $r = 1$

so the series is

$$a + a + a + a + \dots + a + \dots$$

nth partial sum is

$$\underbrace{a + a + \dots + a}_{n \text{ terms}} = na$$

$\boxed{\text{If } a \text{ is } +ve}$

So $\lim_{n\to\infty} S_n = \lim_{n\to\infty} na = +\infty$

$\boxed{\text{If } a \text{ is } -ve}$

So $\lim_{n\to\infty} S_n = \lim_{n\to\infty} na = -\infty$

So if $r=1$, then the geometric series diverges

If $r \neq 1$

$S_n = a + ar + ar^2 + ar^3 + \dots + ar^{n-1}$

$rS_n = ar + ar^2 + ar^3 + ar^4 + \dots + ar^n$

$S_n - rS_n = a - ar^n$

$S_n(1-r) = a(1-r^n)$

$$S_n = \frac{a(1-r^n)}{1-r}, \quad r \neq 1$$

① If $|r|<1$ so $r^n \to 0$ as $n \to \infty$

So $S_n = \frac{a}{1-r}$

② If $|r|>1$ so $|r^n| \to \infty$

So the series diverges

$$a + ar + ar^2 + \dots + ar^{n-1} + \dots \quad s$$

* If $|r| < 1$, the geometric series converges to $\frac{a}{1-r}$, $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$, $|r| < 1$

* If $|r| > 1$ so the series diverge.

Remark: We have determine when (1) a geometric series (2) converges or diverges and to what value.
Often we can determine that a series converges without knowing the value to which it converges

Example:
The geometric series with $a = \frac{1}{9}$ and $r = \frac{1}{3}$ is

$$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \dots = \sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{1}{3}\right)^{n-1}$$

$|r| = \frac{1}{3} < 1$ so the geometric series converges

and it converges to $\frac{a}{1-r} = \frac{\frac{1}{9}}{1-\frac{1}{3}} = \frac{\frac{1}{9}}{\frac{2}{3}} = \frac{1}{6}$

Example:

The series $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 5}{4^n} = 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots$

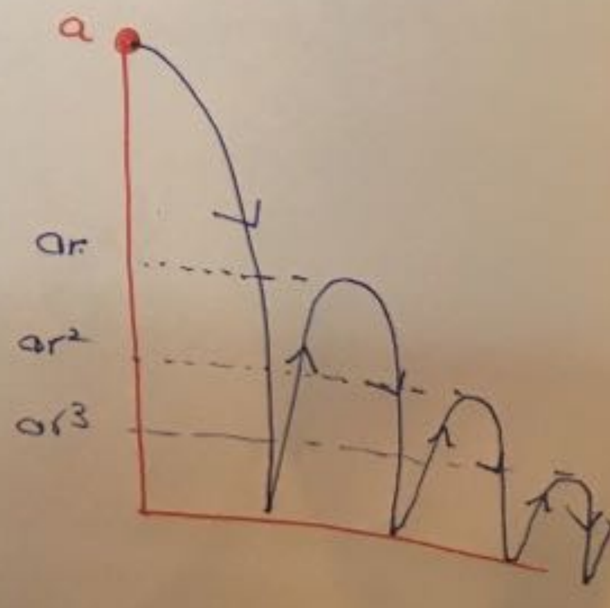
geometric series

$a = 5$

$r = \frac{-1}{4}$, $|r| = \frac{1}{4} < 1$ so the series converges

and it converges to $\frac{a}{1-r} = \frac{5}{1 - -\frac{1}{4}} = \cancel{4}\ \frac{5}{\frac{5}{4}} = 4$

Example: You drop a ball from a meters above a flat surface. Each time the ball hits the surface after falling a distance h, it rebounds a distance rh, where r is positive but less than 1. Find the total distance the ball travels up and down.



So total distance is

$$a + 2ar + 2ar^2 + 2ar^3 + \cdots = a + \frac{2ar}{1-r}$$

$$= a\left(\frac{1+r}{1-r}\right)$$

If $a = 6\,m$ and $r = \frac{2}{3}$

$6 + 8 + 5.333 + 3.555 + 2.3703 + \cdots$

total distance is $30\,m = \cancel{\frac{6}{1-\frac{2}{3}}} = \cancel{\frac{}{\frac{1}{3}}}\ \frac{6(1+\frac{2}{3})}{1-\frac{2}{3}} = 30$

Example:

Express $5.23232323\ldots$ as the ratio of two integers

$$5.232323\ldots = 5 + \frac{23}{100} + \left(\frac{23}{100^2}\right) + \left(\frac{23}{100^3}\right) + \cdots$$

$$= 5 + \frac{23}{100} + \frac{23}{(100)^2} + \frac{23}{(100)^3} + \cdots$$

$$= 5 + \frac{23}{100}\left[1 + \frac{1}{100} + \frac{1}{100^2} + \cdots\right] \qquad a = 1, \quad r = \frac{1}{100}$$

$$= 5 + \frac{23}{100}\left[\frac{1}{1 - \frac{1}{100}} = \frac{1}{\frac{99}{100}}\right]$$

$$= 5 + \frac{23}{100} \cdot \frac{100}{99} = 5 + \frac{23}{99} = \frac{518}{99}$$