10.3 The Integral Test

* Given a series, we want to know whether a series converges or not.
* we just study the series with nonnegative terms. (10.3 + 10.4 + 10.5)

Nondecreasing Partial Sums

$\sum_{n=1}^{\infty} a_n$ is an infinite series with $a_n \geq 0 \ \forall n$.

Each partial sum is greater than or equal to the predecessor because $s_{n+1} = s_n + a_n$

$$s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_n \leq s_{n+1} \leq \cdots$$

Notice that the partial sums form a nondecreasing sequence.

The Monotonic Sequence Th. gives the following result

Corollary of Theorem 6:

A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges iff its partial sums are bounded from above.

Example:

The series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$

Harmonic series

$1 \to 1$

$\frac{1}{2} \to \frac{1}{2}$

$\frac{1}{3} + \frac{1}{4} > \frac{2}{4} = \frac{1}{2}$ (2 terms)

$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{4}{8} = \frac{1}{2}$ (4 terms)

$\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} > \frac{8}{16} = \frac{1}{2}$ (8 terms)

the sum of the next 16 terms is $> \frac{16}{32} = \frac{1}{2}$

In general sum of 2^n terms is greater than $\frac{2^n}{2^{n+1}} = \frac{1}{2}$

so the sequence of partial sums is not bounded from above.

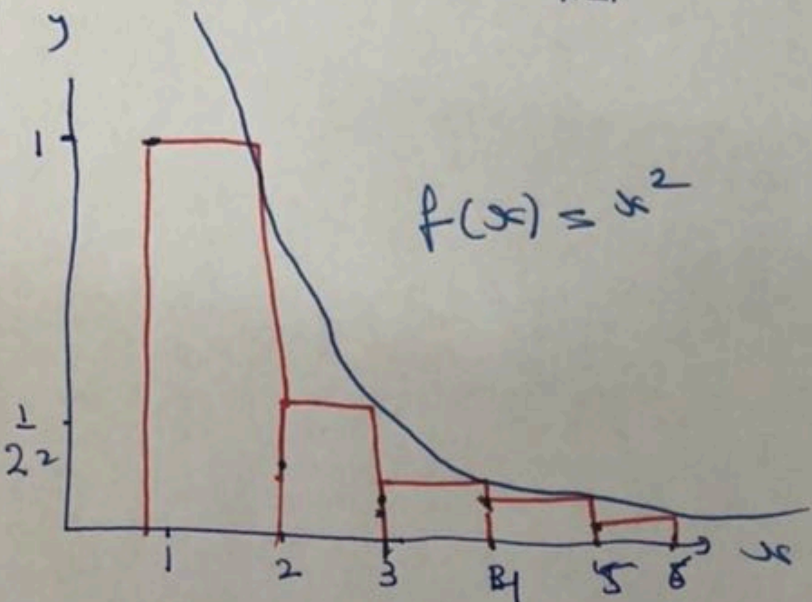
so The harmonic series diverges

The Integral Test

Example: Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{6} + \cdots + \frac{1}{n^2} + \cdots$$

we will compare $\sum_{n=1}^{\infty} \frac{1}{n^2}$ by the Integral $\int_1^{\infty} \frac{1}{x^2}\,dx$



$$S_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots + \frac{1}{n^2}$$

$$= f(1) + f(2) + f(3) + \cdots + f(n)$$

$$< f(1) + \int_1^n \frac{1}{x^2}\,dx$$

$$< 1 + \int_1^{\infty} \frac{1}{x^2}\,dx$$

$$< 1 + 1 = 2$$

8.7

$$\int_1^{\infty} \frac{1}{x^2}\,dx$$

$$= \lim_{a\to\infty} \int_1^{a} \frac{1}{x^2}\,dx$$

$$= \lim_{a\to\infty} -x^{-1} \Big|_1^{a}$$

$$= \lim_{a\to\infty} -\frac{1}{a} + 1$$

$$= 1$$

So $S_n < 2$ so S_n is bounded

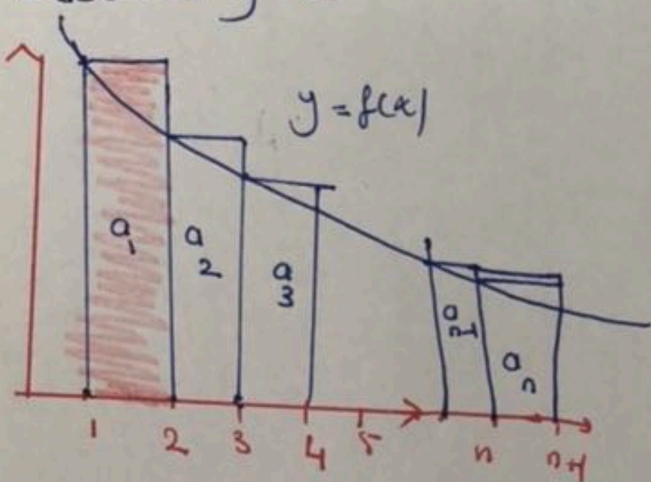
$\rightarrow \sum \frac{1}{n^2}$ converges

Theorem 9 - The Integral Test

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x $\forall\ x \geq N$ (N is +ve integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the Integral $\int_N^{\infty} f(x)\,dx$ both converge or diverge.

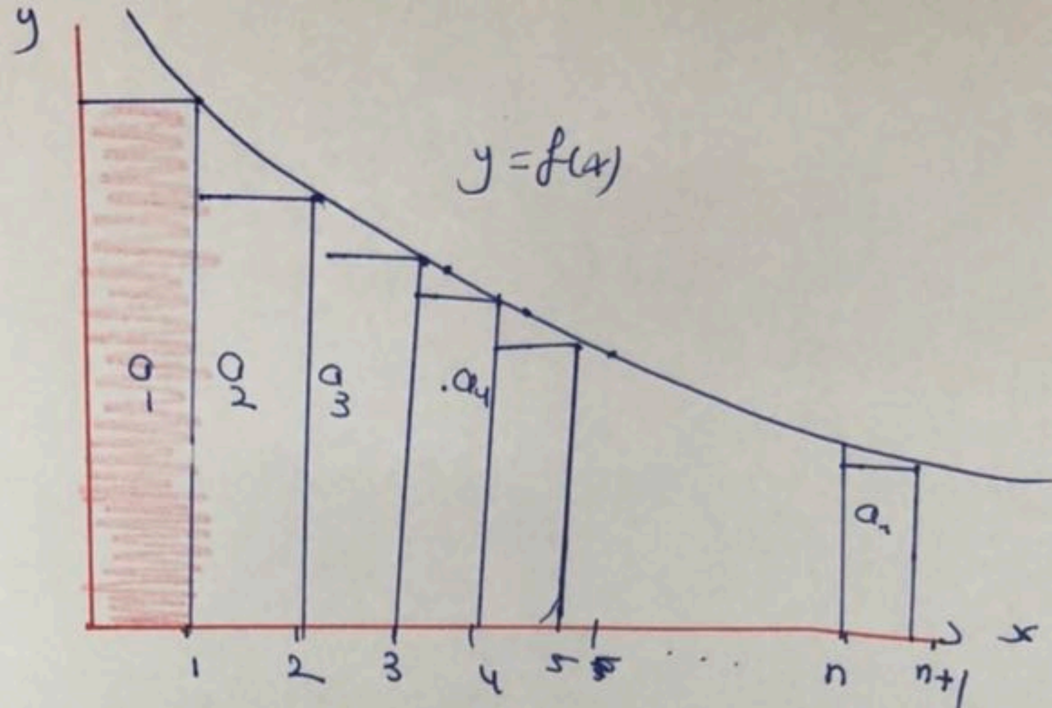
Proof: $N = 1$

f is a decreasing function with $f(n) = a_n$ $\forall\ n$



Notice that the rectangles have more area than that under the curve $y = f(x)$ from $x = 1$ to $n+1$

$$\int_1^{n+1} f(x)\,dx \leq a_1 + a_2 + \cdots + a_n \qquad (1)$$



$$a_1 + a_2 + a_3 + \cdots + a_n \leq a_1 + \int_1^n f(x)\,dx \quad (2)$$

$$\int_1^{n+1} f(x)\,dx \leq a_1 + a_2 + \cdots + a_n \leq a_1 + \int_1^n f(x)\,dx$$

These inequalities hold for each n and continuous to $n \to \infty$

If $\int_1^\infty f(x)\,dx$ is finite so $\sum a_n$ is finite. (look at right hand inequality)

If $\int_1^\infty f(x)\,dx$ is infinite so $\sum a_n$ ~~diverges~~ is infinite (look at left-hand inequality)

so the series and the integral are both finite or infinite

Example: The series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$, $f(x) = \frac{1}{x^2+1}$ +ve , cont, decreasing

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{a\to\infty} \int_1^{a} \frac{1}{x^2+1}\,dx$$

$$= \lim_{a\to\infty} \left[\arctan x\right]_1^{a}$$

$$= \lim_{a\to\infty} \tan^{-1} a - \tan^{-1} 1$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

So the ~~later~~ $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges

[we don't know where it converges to
we don't know its sum]