Sec 10.3: The integral test

• The integral test:

let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, decreasing function of x for all $x \geq N$. Then the series and the integral $\int_N^\infty f(x)\,dx$ both converge or both diverge.

أي أنه إذا كانت a_n موجبة ومتناقصة بعد حد معين و الاقتران سهل التكامل كنلجأ لاختبار التكامل وهو في حال كان التكامل converge تكون series أيضاً converge وفي حال كان diverge تكون series أيضاً converge .

* P - series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$:-

1. If $p > 1$, converges by integral test.
2. If $0 < p \leq 1$, diverges by integral test.
3. If $p \leq 0$; diverges by n-th term test.

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• Determine if the series converge or diverge:

$\boxed{6}$ $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

$\rightarrow f(x) = \dfrac{1}{x(\ln x)^2}$ +ve, continuous,

$$f'(x) = \frac{-\left[x\left(2\ln x \cdot \frac{1}{x}\right) + (\ln x)^2\right]}{x^2 (\ln x)^4}$$

$$= \frac{-2}{x^2(\ln x)^3} - \frac{1}{x^2(\ln x)^2}$$

$f'(x)$ $\longleftarrow \overset{}{\underset{2}{|}}$ - - - - - - - - - - - $\longrightarrow$ $\therefore f(x)$ is decreasing $\forall x \geq 2$

Now $\displaystyle\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{a \to \infty} \int_2^{a} \frac{dx}{x(\ln x)^2}$

To find $\displaystyle\int \frac{dx}{x(\ln x)^2}$

let $u = \ln x \longrightarrow du = \dfrac{dx}{x}$

$$\int \frac{dx}{x(\ln x)^2} = \int \frac{du}{u^2} = \frac{-1}{u} = \frac{-1}{\ln x}$$

$$\int_2^\infty \frac{dx}{x(\ln x)^2} = \lim_{a\to\infty} \frac{-1}{\ln x}\Big|_2^a$$

$$= \lim_{a\to\infty} \frac{-1}{\cancel{\ln a}_{\infty}} + \frac{1}{\ln 2} = \frac{1}{\ln 2}$$

$\therefore \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges by integral test.

$\boxed{13}$ $\sum_{n=1}^{\infty} \frac{n}{n+1}$

$\longrightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$

$\therefore \sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges by n th term test.

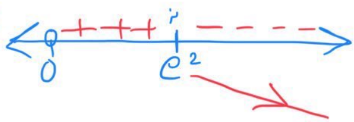
$\boxed{20}$ $\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$

$\longrightarrow f(x) = \frac{\ln x}{\sqrt{x}}$ +ve, continuous

$$f'(x) = \frac{\sqrt{x} \cdot \frac{1}{x} - \ln x \cdot \frac{1}{2\sqrt{x}}}{(\sqrt{x})^2}$$

$$= \frac{1}{x\sqrt{x}} - \frac{\ln x}{2x\sqrt{x}} = \frac{1}{x\sqrt{x}}\left(1 - \frac{\ln x}{2}\right)$$

$\ln x > 2 \quad \forall x > e^2$

$f'(x)$


Now $\displaystyle\int_2^{\infty} \frac{\ln x}{\sqrt{x}}\,dx = \lim_{a\to\infty} \int_2^{a} \frac{\ln x}{\sqrt{x}}\,dx$

to find $\displaystyle\int \frac{\ln x}{\sqrt{x}}\,dx$

let $\ln x = u \longrightarrow x = e^u$

$$\frac{dx}{x} = du$$

$$\int \frac{\ln x}{\sqrt{x}}\,dx = \int \frac{u}{\sqrt{e^u}} \cdot e^u\,du$$

$$= \int u e^{\frac{1}{2}u}\,du$$

$$= 2ue^{\frac{1}{2}u} - 4e^{\frac{1}{2}u}$$

u	$e^{\frac{1}{2}u}$
1	$2e^{\frac{1}{2}u}$
0	$4e^{\frac{1}{2}u}$

$$= 2(\ln x)\sqrt{x} - 4\sqrt{x}$$

$$\therefore \int_2^{\infty} \frac{\ln x}{\sqrt{x}}\,dx = \lim_{a\to\infty} 2\ln x\sqrt{x} - 4\sqrt{x}\,\Big|_2^a$$

$$= \lim_{a\to\infty}\left(2\sqrt{a}\ln a - 4\sqrt{a}\right) - \left(2\sqrt{2}\ln 2 - 4\sqrt{2}\right)$$

$$= \lim_{a\to\infty} 2\sqrt{a}(\ln a - 2) - (2\sqrt{2} - 4\sqrt{2})$$

$$= \infty \qquad \therefore \int_2^\infty \frac{\ln x}{\sqrt{x}}\,dx \text{ diverges}$$

$\therefore \sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$ diverges by integral test.

$\boxed{22}$ $\sum_{n=1}^{\infty} \frac{5^n}{4^n+3}$

$$\longrightarrow \lim_{n\to\infty} \frac{5^n}{4^n+3} = \lim \frac{5^n \ln 5}{4^n \ln 4}$$

$$= \lim_{n\to\infty} \left(\frac{5}{4}\right)^n \left(\frac{\ln 5}{\ln 4}\right)$$

$$= \infty \neq 0$$

$\therefore \sum_{n=1}^{\infty} \frac{5^n}{4^n+3}$ diverges by n-th term test

$\boxed{28}$ $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$

$$\longrightarrow \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \neq 0$$

$\therefore \sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$ diverges

32 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(1+\ln^2 n)}$

$\longrightarrow f(x) = \dfrac{1}{x(1+\ln^2 x)}$ +ve, continuous, decreasing, integrable.

Now $\displaystyle\int_1^{\infty} \frac{dx}{x(1+\ln^2 x)} = \lim_{a\to\infty} \int_1^{a} \frac{dx}{x(1+\ln^2 x)}$

let $u = \ln x \longrightarrow du = \frac{1}{x}\,dx$

$$\int \frac{dx}{x(1+\ln^2 x)} = \int \frac{du}{1+u^2} = \tan^{-1} u = \tan^{-1}(\ln x)$$

$$\therefore \int_1^{\infty} \frac{dx}{x(1+\ln^2 x)} = \lim_{a\to\infty} \tan^{-1}(\ln x)\Big|_1^a$$

$$= \lim_{a\to\infty} \tan^{-1}(\ln a) - \tan^{-1}(\ln 1)$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$\longrightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1}{n(1+\ln^2 n)}$ converges by integral test.

38 $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

$$\longrightarrow f(x) = \frac{x}{x^2+1} \quad +\text{re}, \text{ cont. } \forall x \geq 1$$

$$f'(x) = \frac{(x^2+1)(1) - (x)(2x)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$$

$$f'(x) \quad \underset{1}{\longleftrightarrow} \quad \text{decreasing. } \forall x \geq 1$$

$$\text{Now } \int_1^\infty \frac{x}{x^2+1}\,dx = \lim_{a\to\infty} \int_1^a \frac{x}{x^2+1}\,dx$$

$$\frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1)$$

$$\therefore \int_1^\infty \frac{x}{x^2+1}\,dx = \lim_{a\to\infty} \frac{1}{2}\ln(x^2+1)\Big|_1^a$$

$$= \lim_{a\to\infty} \frac{1}{2}\ln(a^2+1) - \frac{1}{2}\ln 2$$

$$= \infty \qquad \therefore \int_1^\infty \frac{x}{x^2+1}\,dx \text{ diverges}$$

So $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges by integral test.

40 $\sum_{n=1}^{\infty} \operatorname{sech}^2 n$

$\longrightarrow \operatorname{sech}^2 x = \frac{1}{\cosh^2 x} = \left(\frac{2}{e^x + e^{-x}}\right)^2$

+ve, dec., cont. $\forall x > 1$

Now $\int_1^{\infty} \operatorname{sech}^2 x \, dx = \lim_{a\to\infty} \int_1^{a} \operatorname{sech}^2 x \, dx$

$= \lim_{a\to\infty} \tanh x \Big|_1^a$

$= \lim_{a\to\infty} \tanh a - \tanh 1$

$= 1 - \tanh 1$

$\therefore \int_1^{\infty} \operatorname{sech}^2 x \, dx$ converges so

$\sum_{n=1}^{\infty} \operatorname{sech}^2 n$ converges by integral test.

42 For what values of a, if any, do the series converge?

$$\sum_{n=3}^{\infty} \left(\frac{1}{n-1} - \frac{2a}{n+1}\right)$$

$$\longrightarrow \int_3^{\infty}\left(\frac{1}{x-1}-\frac{2a}{x+1}\right)dx$$

$$=\lim_{b\to\infty}\int_3^{b}\left(\frac{1}{x-1}-\frac{2a}{x+1}\right)dx$$

$$=\lim_{b\to\infty}\left(\ln|x-1|-2a\ln|x+1|\right)\Big|_3^b$$

$$=\lim_{b\to\infty}\ln\left(\frac{x-1}{(x+1)^{2a}}\right)\Big|_3^b$$

$$=\lim_{b\to\infty}\ln\left[\frac{b-1}{(b+1)^{2a}}\right]-\ln\frac{2}{4^{2a}}$$

$$=\ln\lim_{b\to\infty}\frac{1}{2a(b+1)^{2a-1}}-\ln\left(\frac{2}{4^{2a}}\right)$$

$$=\begin{cases}0-\ln\left(\frac{2}{4^{2a}}\right) & ; = \frac{1}{2}\\ \infty & ; a<\frac{1}{2}\end{cases}$$

if $a>\frac{1}{2}$; $\sum_{n=3}^{\infty}\left(\frac{1}{n-1}-\frac{2a}{n+1}\right)$

$-$ve, the integral test doesn't apply.

but the series diverges ("harmonic series").