Example: Show that the p series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

P is real constant converges if $P>1$ and diverges if $P \le 1$

Solution:

If $P>1$, $f(x) = \frac{1}{x^P}$ +ve, decreasing function

$$\int_1^{\infty} \frac{1}{x^P}\, dx = \int_1^{\infty} x^{-P}\, dx = \lim_{b\to\infty} \int_1^b x^{-P}\, dx$$

$$= \lim_{b\to\infty} \left. \frac{x^{-P+1}}{-P+1} \right|_1^b$$

$$= \frac{1}{-P+1} \lim_{b\to\infty} \left(b^{1-P} - 1 \right)$$

$$= \frac{1}{1-P} \lim_{b\to\infty} \left[\frac{1}{b^{P-1}} - 1 \right]$$

$$= \frac{1}{1-P} (0-1) = \frac{1}{P-1}$$

So $\sum_{n=1}^{\infty} \frac{1}{n^P}$, $P>1$ converges by the Integral Test

If $p < 1$

$$\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{1-p} \lim_{b\to\infty}\left(b^{1-p} - 1\right) = \infty$$

So $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $p<1$ diverges by the Integral Test.

If $p=1$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$$

which is the divergent harmonic serie

So $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $p>1$ converges and if $p\le 1$ it diverges

Error Estimation:

If $\sum a_n$ is shown to be convergent by the Integral Test, we may estimate size of the remainder R_n between the total sum S of the series and its nth partial sums S_n.

$$R_n = S - S_n = \left(a_1 + a_2 + a_3 + \cdots + a_n + \cdots\right) - \left(a_1 + a_2 + \cdots + a_n\right)$$

$$= a_{n+1} + a_{n+2} + \cdots + a_{n+100} + \cdots$$

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Bounds for Remainder in the Integral Test

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx$$

$$\int_{n+1}^{\infty} f(x)\,dx \le S - S_n \le \int_{n}^{\infty} f(x)\,dx$$

$$\boxed{S_n + \int_{n+1}^{\infty} f(x)\,dx \le S \le S_n + \int_{n}^{\infty} f(x)\,dx}$$

Ex Estimate the sum of the series $\sum \frac{1}{n^2}$ using the above inequality and $n=10$

$$\int_{n}^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left.\frac{-1}{x}\right|_{n}^{b} = \lim_{b\to\infty} \frac{-1}{b} + \frac{1}{n} = \frac{1}{n}$$

$$S_n + \frac{1}{n+1} \le S \le S_n + \frac{1}{n}$$

$$S_{10} + \frac{1}{11} \le S \le S_{10} + \frac{1}{10}$$

$$S_{10} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{100} \approx 1.54977$$

$$1.64068 \le S \le 1.64977 \rightarrow S \cong 1.6453$$

(midpoint of the last interval)

(8) $\sum_{n=2}^{\infty} \frac{\ln n^2}{n}$

$\lim_{n\to\infty} \frac{\ln n^2}{n} = 0$ nth term test fails

$f(x) = \frac{\ln x^2}{x}$ cont

+ve

decreasing

$f(x) = \frac{2\ln x}{x}$

$f'(x) = \frac{x \cdot \frac{2}{x} - 2\ln x}{x^2} = \frac{2 - 2\ln x}{x^2} = \frac{2(1-\ln x)}{x^2}$

$1 - \ln x = 0 \rightarrow \ln x = 1 \rightarrow x = e.$

+ - - - - -

2 e

$f(x)$ is decreasing for $x \geq 3$

so we can apply the Integral Test on $[3, \infty)$

$\int_3^{\infty} \frac{2\ln x}{x}\,dx = \lim_{b\to\infty} \int_3^{b} \frac{2\ln x}{x}\,dx = \lim_{b\to\infty} 2\ln x \Big|_3^b$

$= 2\lim_{b\to\infty}(\ln b - \ln 3)$

$= \infty$

$\int_3^{\infty} \frac{\ln x^2}{x}\,dx$ diverges so by Integral test also

$\sum_{n=3}^{\infty} \frac{\ln n^2}{n}$ diverges $\Rightarrow \sum_{n=2}^{\infty} \frac{\ln n^2}{n}$ diverges

since $\sum_{n=2}^{\infty} \frac{\ln n^2}{n} = \frac{\ln 4}{2} + \sum_{n=3}^{\infty} \frac{\ln n^2}{n}$

(26) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)}$

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$\lim_{x \to \infty} \frac{1}{\sqrt{x}(\sqrt{x}+1)} = 0$ so nth term test diverges

$f(x) = \frac{1}{\sqrt{x}(\sqrt{x}+1)}$ +ve decreasing cont. check!

$\int_1^{\infty} \frac{dx}{\sqrt{x}(\sqrt{x}+1)} = \lim_{b \to \infty} \int_1^b \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$

$\int \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$

Let $u = \sqrt{x}+1$

$du = \frac{dx}{2\sqrt{x}}$

$I = \int \frac{2\sqrt{x}\, du}{\sqrt{x}\, u}$

$= 2\ln|u| + c$

$= 2\ln(\sqrt{x}+1) + c$

$= \lim_{b \to \infty} 2\ln|\sqrt{x}+1| \Big|_1^b$

$= \lim_{b \to \infty} 2\ln|\sqrt{b}+1| - 2\ln 2$

$= \infty$

So $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)}$ diverges by ~~the~~ Integral test

34 $\sum_{n=1}^{\infty} n \tan \frac{1}{n}$

$$\lim_{n\to\infty} n \tan \frac{1}{n}$$

$$= \lim_{n\to\infty} \frac{\tan \frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{\sec^2 \frac{1}{n} \cdot \frac{-1}{n^2}}{\frac{-1}{n^2}} = 1 \neq 0$$

So $\sum_{n=1}^{\infty} n \tan \frac{1}{n}$ diverges by nth term test

10.4 Comparison Tests

Direct Comparison Test

Let $\sum a_n$, $\sum c_n$, and $\sum d_n$ be series with nonnegative terms. Suppose that for some integer N

$$d_n \le a_n \le c_n, \text{ for all } n > N$$

a) If $\sum c_n$ converges, then $\sum a_n$ also converges

b) If $\sum d_n$ ~~converges~~ diverges, then $\sum a_n$ also diverges

Example:

① $\sum_{n=1}^{\infty} \frac{5}{5n-1}$

$\lim_{n\to\infty} \frac{5}{5n-1} = 5 \neq 0$

so $\sum_{n=1}^{\infty} \frac{5}{5n-1}$ diverges using nth term test

also we can use D.C.T

$$\frac{5}{5n-1} = \frac{5}{5} \cdot \frac{1}{n-\frac{1}{5}} = \frac{1}{n-\frac{1}{5}} > \frac{1}{n}$$

so $\sum_{n=1}^{\infty} \frac{5}{5n-1} > \sum_{n=1}^{\infty} \frac{1}{n}$, $\sum_{n=1}^{\infty} \frac{1}{n}$ harmonic series

so $\sum \frac{5}{5n-1}$ diverges using D.C.T with $\sum \frac{1}{n}$

(b) $\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$

$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$

$1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + 1 + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots$

① so $\sum_{n=0}^{\infty} \frac{1}{n!} < 1 + \sum_{n=0}^{\infty} \frac{1}{2^n}$

② $1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{1-\frac{1}{2}} = 3$

So the series $1 + \sum_{n=0}^{\infty} \frac{1}{2^n}$ converges

so $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges by D.C.T

also $n! = n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1 > n(n-1)$

$\frac{1}{n!} \cancel{<} \frac{1}{n(n-1)}$

$\cancel{\sum_{0}^{\infty} \frac{1}{n!} < \sum_{0}^{\infty} \frac{1}{n(n-1)}}$

$\cancel{\sum_{n=0}^{\infty} \frac{1}{n(n-1)} = \frac{a}{n} + b}$