10.4 Comparison Tests

1 Direct Comparison Test

If $\sum a_n$, $\sum c_n$ and $\sum d_n$ are series with nonnegative terms and

$$d_n \leq a_n \leq c_n \quad \text{for all } n > N$$

a) If $\sum c_n$ converges, then $\sum a_n$ also converges

b) If $\sum d_n$ diverges, then $\sum a_n$ also diverges

Example:

ⓐ $\displaystyle\sum_{n=1}^{\infty} \frac{5}{5n-1}$

$\displaystyle\lim_{n \to \infty} \frac{5}{5n-1} = 0$ (nth term test fails)

$$\frac{5}{5n-1} = \frac{5}{5(n-\frac{1}{5})} = \frac{1}{n-\frac{1}{5}} > \frac{1}{n}$$

So $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n-\frac{1}{5}} > \sum_{n=1}^{\infty} \frac{1}{n}$

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic)

So $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n-\frac{1}{5}}$ also diverges by D.C.T

(b) $\sum_{n=0}^{\infty} \frac{1}{n!} = 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$

$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots$

$1+\sum_{n=0}^{\infty} \frac{1}{2^n} = 1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots$

(1) $\sum_{n=0}^{\infty} \frac{1}{n!} \leq 1+\sum_{n=0}^{\infty} \frac{1}{2^n}$

(2) $1+\sum_{n=0}^{\infty} \frac{1}{2^n} = 1+\frac{1}{1-\frac{1}{2}} = 1+\frac{1}{\frac{1}{2}} = 3$

So the series $1+\sum_{n=0}^{\infty} \frac{1}{2^n}$ converges to 3

and $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges by D.C.T

(c) $5 + \frac{2}{3} + \frac{1}{7} + 1 + \frac{1}{2+\sqrt{1}} + \frac{1}{4+\sqrt{2}} + \dots + \frac{1}{2^n+\sqrt{n}} + \dots$

$\lim_{n\to\infty} \frac{1}{2^n+\sqrt{n}} = 0$ (nth term test fails)

$$\sum_{n=0}^{\infty} \frac{1}{2^n+\sqrt{n}} < \sum_{n=0}^{\infty} \frac{1}{2^n}$$

Ignore the first three terms

$$1 + \frac{1}{2+\sqrt{1}} + \frac{1}{4+\sqrt{2}} + \frac{1}{8+\sqrt{8}} + \dots \leq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

So $\sum \frac{1}{2^n+\sqrt{n}}$ converges by D.C.T

Note that $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is a convergent geometric series

Limit Comparison Test LCT

suppose $a_n > 0$, $b_n > 0$ $\forall\ n \geq N$

- If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge

- If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ conv.

③ If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges

Example:

a) $\sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series)

$$\lim_{n\to\infty} \frac{\frac{2n+1}{n^2+2n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{2n^2+n}{n^2+2n+1} = 2 \text{ so both diverges}$$

so $\sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$ diverges by L.C.T with $\sum_{1}^{\infty} \frac{1}{n}$

b) $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

$\sum_{n=1}^{\infty} \frac{1}{2^n}$ Conv. "geometric, $r=\frac{1}{2}<1$"

$$\lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{2^n}{2^n-1} = \lim_{n\to\infty} \frac{1}{1-\left(\frac{1}{2}\right)^n} = 1$$

so both converges, $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ conv. by L.C.T with $\sum_{n=1}^{\infty} \frac{1}{2^n}$

(c) $\sum_{n=2}^{\infty} \frac{1+n\ln n}{n^2+5}$

$\frac{1+n\ln n}{n^2+5}$ behaves like $\frac{n\ln n}{n^2} = \frac{\ln n}{n^2}$, for $n\to\infty$

and $\ln n > 1$ $\forall\, n \geq 2$

So $\frac{1+n\ln n}{n^2+5}$ for large n is greater than $\frac{1}{n}$

$\sum_{n=2}^{\infty} \frac{1}{n}$ diverges ".

$$\lim_{n\to\infty} \frac{\frac{1+n\ln n}{n^2+5}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n+n^2\ln n}{n^2+5} = \infty$$

So $\sum \frac{1+n\ln n}{n^2+5}$ diverges

Example: Does $\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$ converge ?

$$\frac{\ln n}{n^{\frac{3}{2}}} < \frac{n^{\frac{1}{4}}}{n^{\frac{3}{2}}} < \frac{1}{n^{5/4}}$$

$$\lim_{n\to\infty} \frac{\dfrac{\ln n}{n^{\frac{3}{2}}}}{\dfrac{1}{n^{5/4}}} = \lim_{n\to\infty} \frac{\ln n}{n^{\frac{1}{4}}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{4} n^{-\frac{3}{4}}} = \lim_{n\to\infty} \frac{4}{n^{\frac{1}{4}}} = 0$$

and $\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$ (P-series, $P>1$) so converges

so $\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$ conv. by L.C.T with $\sum \frac{1}{n^{5/4}}$

Notes:

$$\frac{\ln n}{n^{3/2}} < \frac{n^{c}}{n^{\frac{3}{2}}} = \frac{1}{n^{\frac{3}{2}-c}}$$

$$\frac{3}{2} - c > 1$$

$$\frac{3}{2} - 1 > c$$

$$\boxed{\frac{1}{2} > c}$$

10.4 10 + 22 + 36 + 46

(10) $\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n^2+2}}$

for large n, it behave like

$\sqrt{\frac{n}{n^2}} = \sqrt{\frac{1}{n}} = \frac{1}{\sqrt{n}}$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (P-series $P = \frac{1}{2} < 1$)

$$\lim_{n\to\infty} \frac{\sqrt{\frac{n+1}{n^2+2}}}{\sqrt{\frac{1}{n}}} = \lim_{n\to\infty} \sqrt{\frac{n^2+n}{n^2+2}} = 1$$

So $\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n^2+2}}$ ~~converge~~ diverge by L.C.T

(22) $\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$

for large n it acts like

$\frac{n}{n^2\sqrt{n}} = \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ conv. (P-series $P = \frac{3}{2} > 1$)

$$\lim_{n\to\infty} \frac{\frac{n+1}{n^2\sqrt{n}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n+1}{n} = 1$$

$\sum \frac{n+1}{n^2\sqrt{n}}$ conv. by L.C.T

So both converges.