— Sec 10.7 : Power Series

* Def : A power series about $x=0$ is a series of the form:-

$$\sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n$$

A power series about $x=a$ is a series of the form:-

$$\sum_{n=0}^{\infty} C_n (x-a)^n = C_0 + C_1 (x-a) + \cdots$$

* Thm : The convergence theorem for power series:-

If the power series converges at $x=c$, then it converges absolutely for all x with $|x|<|c|$. If the series diverges at $x=d$, then it diverges for all x with $|x|>|d|$.

* How to test a power series for convergence:-

1. Use the ratio test (or n-th root test) to find the interval where the series converges absolutely.
2. If the interval of convergence is finite, test for convergence at each endpoint. (Use a C.T, Integral test, alternating series test).

③ Otherwise, the series diverges.

* The term by term differentiation theorem:-

$$f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$$

$$\rightarrow f'(x) = \sum_{n=1}^{\infty} n C_n (x-a)^{n-1}$$

⋮

* The term by term integration:-

$$f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$$

$$\rightarrow \int f(x)dx = \sum_{n=0} \frac{C_n (x-a)^{n+1}}{n+1}$$

- Exercises: page 582

4 a) find the series' radius and interval of convergence.

$$\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$$

b) For what values of x does the series converges absolutely, conditionally?

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(3X-2)^{n+1}}{n+1}\cdot\frac{n}{(3X-2)^n}\right|$$

$$= \lim_{n\to\infty}\frac{n}{n+1}|3X-2|$$

$$= |3X-2| < 1$$

$$\longrightarrow \quad -1 < 3x-2 < 1$$
$$+2 \qquad +2 \qquad +2$$

$$\frac{1}{3} < \frac{3X}{3} < \frac{3}{3}$$

$$\therefore \frac{1}{3} < X < 1$$

↳ converges absolutely.

at $X=\frac{1}{3}$: نعوض في الseries الأصلية

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$ converges conditionally «alternating p-series»

at $X=1$:- نعوض ال $X=1$ في series الأصلية

$$\sum_{n=1}^{\infty}\frac{1}{n}$$ diverges «p-series with $p=1$»

so the series $\sum_{n=1}^{\infty}\frac{(3X-2)^n}{n}$
- → converges absolutely $(\frac{1}{3}, 1)$
- → converges conditionally $X=\frac{1}{3}$
- → diverges $(-\infty, \frac{1}{3}) \cup [1, \infty)$

the radius of convergence: $\frac{1-\frac{1}{3}}{2} = \frac{1}{3}$

center: $\frac{1+\frac{1}{3}}{2} = \frac{2}{3}$

12 $\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$

$\longrightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{3^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n x^n}\right|$

$= \lim_{n\to\infty} \frac{3}{n+1} |x| < 1$

$= 0 \cdot |x| < 1$ always « for all x »

$\therefore$ the series converges absolutely for all x : $(-\infty, \infty)$

R.O.C $= \infty$

center $= 0$

14 $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^3 3^n}$

$\longrightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-1)^{n+1}}{(n+1)^3 3^{n+1}} \cdot \frac{n^3 3^n}{(x-1)^n}\right|$

$= \lim_{n\to\infty} |x-1| \frac{n^3}{3(n+1)^3}$

$= \frac{1}{3}|x-1| < 1$

$\longrightarrow \quad -3 < x-1 < 3$
$\quad\quad\quad +1 \quad\; +1 \quad\; +1$

$-2 < x < 4$ converges absolutely.

• at $x = -2$: تعوض 2- في series الأصلية.

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n^3\, 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$$ converges absolutely «alternating p-series «p=3»»

at $x = 4$:

$$\sum_{n=1}^{\infty} \frac{(3)^n}{n^3\, 3^n} = \sum_{n=1}^{\infty} \frac{1}{n^3}$$ ∴ converges «p-series with p = 3»

∴ at $x = 4$ converges absolutely.

Now: IOC = $[-2, 4]$

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^3\, 3^n}$$

→ converges absolutely $-2 \leq x \leq 4$

→ converges conditionaly no values.

→ diverges $(-\infty, -2) \cup (4, \infty)$

R.O.C = 3

center = 1

23 $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n x^n$ n-th root test

$\rightarrow \lim_{n\to\infty} \sqrt[n]{\left|\left(1+\frac{1}{n}\right)^n x^n\right|} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)|x|$

$= |x| < 1$ converges abs.

at $x=-1$:-

$\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n (-1)^n$ alternating series :-

$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \neq 0$

$\therefore$ diverges by n-th term test.

at $x=1$:-

$\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$ by nth term test.

$\lim \left(1+\frac{1}{n}\right)^n = e \neq 0$

$\therefore$ diverges by nth term test.

Now: $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n x^n$ →
- conv. abs. $-1<x<1$
- conv. cond $\times$.
- diverges $(-\infty,-1] \cup [1,\infty)$

R.O.C $= 1$, center $= 0$

29 $\sum_{n=2}^{\infty} \frac{x^n}{n(\ln n)}$ «by ratio test»

$\rightarrow \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)\ln(n+1)} \cdot \frac{n \ln n}{x^n} \right| = |x| \lim_{n\to\infty} \frac{n \ln n}{(n+1)\ln(n+1)} = |x| < 1$

$\therefore -1 < x < 1$ converges absolutely.

at $x = 1$: $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ «by integral test» +ve, dec., cont.

$\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{a\to\infty} \int_2^{a} \frac{1}{x \ln x} dx = \lim_{a\to\infty} \ln(\ln x) \Big|_2^a = \infty$ diverges.

$\therefore \sum \frac{1}{n \ln n}$ diverges.

at $x = -1$: $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converges cond.

$\therefore$ R.O.C $= 1$
center $= 0$
IOC $= [-1, 1)$

32 $\sum_{n=1}^{\infty} \frac{(3x+1)^{n+1}}{2n+2}$

$\rightarrow \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(3x+1)^{n+2}}{2(n+1)+2} \cdot \frac{2n+2}{(3x+1)^{n+1}} \right|$

$= \lim_{n\to\infty} |3x+1| \frac{2n+2}{2n+4}$

$= |3x+1| < 1$ conv. abs.

$\therefore -1 < 3x+1 < 1$

$-2 < 3x < 0 \longrightarrow \frac{-2}{3} < x < 0$ conv. abs.

at $x = \frac{-2}{3}$: نعوض

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+2}$ alternating series.

$\lim_{n\to\infty} U_n = \lim \frac{1}{2n+2} = 0$

$U_n = \frac{1}{2n+2}$ tre, decreasing.

$\therefore \sum \frac{(-1)^{n+1}}{2n+2}$ converges, Now want to check $\sum \left|\frac{(-1)^{n+1}}{2n+2}\right|$

let $a_n = \frac{1}{2n+2}$, $b_n = \frac{1}{n}$ «L.C.T»

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim \frac{n}{2n+2} = \frac{1}{2}$ and $\sum \frac{1}{n}$ diverges $\therefore \sum \frac{1}{2n+2}$

diverges by L.C.T.

$\therefore$ at $X = \frac{-2}{3}$ conv. cond.

at $X = 0$:-

$\sum_{n=1}^{\infty} \frac{1}{2n+2}$ diverges. by L.C.T with $\frac{1}{n}$.

$\therefore \sum_{n=1}^{\infty} \frac{(3X+1)^{n+1}}{2n+2}$

→ conv. absolutely $\frac{-2}{3} < X < 0$

→ con cond. $X = \frac{-2}{3}$.

→ diverges $(-\infty, -\frac{2}{3}) \cup [0, \infty)$

R.O.C $= \frac{1}{3}$.

center $= \frac{-1}{3}$

40 Find the serie's radius convergence:-

$$\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2} x^n$$

by nth root test

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}|x|^n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n |x|$$

$$= \lim\left(\frac{n+1-1}{n+1}\right)^n |x|$$

$$= \lim\left(1-\frac{1}{n+1}\right)^n |x|$$

$$= e^{-1}|x| < 1$$

$\therefore$ $-1 < \frac{1}{e}x < 1 \longrightarrow -e < x < e$.

$\longrightarrow$ R.OC $= e$

46 $\sum_{n=0}^{\infty}(\ln x)^n$, find the I.O.C and the sum of the series.

$$\sum_{n=0}^{\infty}(\ln x)^n = 1+\ln x+(\ln x)^2+\cdots$$

geometric series converges if $-1<r<1$

$a=1$, $r=\ln x$

$-1<\ln x<1 \longrightarrow e^{-1}<x<e$ converges abs.

otherwise, diverges. the sum of the series is

$$\frac{a}{1-r} = \frac{1}{1-\ln x}$$

$\boxed{49}$ For what values of x does the series

$$1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \cdots + \left(\frac{-1}{2}\right)^n (x-3)^n + \cdots$$

converge? what is its sum? what series do you get if you differentiate the given series term by term? For what values of x does the new series converges? what is its sum?

$$1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 - \cdots + \left(\frac{-1}{2}\right)^n (x-3)^n + \cdots$$

geometric series $a = 1$, $r = \frac{-1}{2}(x-3)$

converges if $\quad -1 < r < 1$

$$-1 < \frac{-1}{2}(x-3) < 1$$

$$-2 < -x + 3 < 2$$

$$-3 \qquad -3 \qquad -3$$

$$-5 < -x < -1 \longrightarrow 1 < x < 5$$

$\therefore$ if $1 < x < 5$, the series $\sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^n (x-3)^n$

converges to $\dfrac{1}{1 - \frac{1}{2}(x-3)}$

$$= \frac{1}{x-1}$$

$\therefore\ 1-\frac{1}{2}(x-3)+\frac{1}{4}(x-3)^2+\cdots=\frac{2}{x-1}$; if $1<x<5$

$f'(x)=\frac{-1}{2}+\frac{1}{2}(x-3)-\cdots\cdots=\frac{-2}{(x-1)^2}$ if $1<x<5$

$\therefore\ \sum_{n=1}^{\infty}\left(\frac{-1}{2}\right)^n n(x-3)^{n-1}=\frac{-2}{(x-1)^2}\ \forall\ 1<x<5$