# 10.8: Taylor and Maclaurin series

**Definition**

Let $f$ be a function with derivatives of all orders throughout some interval containing $a$ as an interior point. Then the Taylor series generated by $f$ at $x = a$ is

$$\sum \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!} + \cdots$$

The ~~Maclarine~~ Maclaurin series generated by $f$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)\, x^k}{k!} = f(0) + f'(0)x + \frac{f''(0)(x)^2}{2!} + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

[It's the same as Taylor series generated by $f$ at $x = 0$]

**Ex** Find the Taylor series generated by $f(x) = \frac{1}{x}$ at $a = 2$

$\boxed{f(x) = x^{-1}}$, $\boxed{f'(x) = -x^{-2}}$, $\boxed{f''(x) = 2!\,x^{-3}}$, ..., $\boxed{f^{(n)}(x) = (-1)^n n!\, x^{-(n+1)}}$

$\boxed{f(2) = \frac{1}{2}}$, $\boxed{f'(2) = -\frac{1}{4}}$, $\boxed{f''(2) = \frac{2!}{8} = \frac{1}{4}}$, $f'''(2) =$ , $\cancel{f^{(n)}(2) =}$

$$\boxed{f^{(n)}(2) = \frac{(-1)^n n!}{2^{n+1}}}$$

Taylor series of $f(x)$ around 2

$$f(2) + f'(2)(x-2) + \frac{f''(2)(x-2)^2}{2!} + \dots + \frac{f^{(n)}(2)(x-2)^n}{n!} + \dots$$

$$= \frac{1}{2} + -\frac{1}{4}(x-2) + \frac{1\,(x-2)^2}{4\cdot 2!} + \dots + \frac{(-1)^n(x-2)^n}{2^{n+1}} + \dots$$

$$= \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} + \dots + (-1)^n\frac{(x-2)^n}{2^{n+1}} + \dots$$

Does this series converge to $\frac{1}{x}$?

Geometric

$a = \frac{1}{2}$, $r = \frac{-(x-2)}{2}$, It converges absolutely for

$|r| = \left|\frac{x-2}{2}\right| < 1 \rightarrow |x-2| < 2 \rightarrow -2 < x-2 < 2$

$0 < x < 4$

and its sum is
$$\frac{\frac{1}{2}}{1 - \frac{-(x-2)}{2}} = \frac{\frac{1}{2}}{\frac{2+x-2}{2}} = \frac{1}{x}$$

So the Taylor series generated by $f(x) = \frac{1}{X}$ at $a=2$ converges to $\frac{1}{x}$ for $|x-2| < 2$ or $0 < x < 4$

## Taylor Polynomial

The linearization of a differentiable function $f$ at a point $a$ is the polynomial of degree one, given by

$$P_1(x) = f(a) + f'(a)(x-a)$$

* the linearization is used to approximate $f(x)$ at values of $x$ near $a$.

* If $f$ has derivatives of higher order at $a$, it has as well higher order polynomial approximations

* These polynomials are called the Taylor Polynomials

**Def**

Let $f$ be a function with derivatives of order $k$ for $k = 1, 2, \dots, N$ in some interval containing $a$ as an interior point. Then for any integer $n$ from 0 through $N$, the Taylor polynomial of order $n$ generated by $f$ at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Example: Find the Taylor series and the Taylor Polynomials generated by $f(x) = e^x$ at $x = 0$

Solution:

$f(x) = e^x$, $f'(x) = e^x$, $f''(x) = e^x$, ..., $f^{(n)}(x) = e^x$

$f(0) = 1$, $f'(0) = 1$, $f''(0) = 1$, ..., $f^{(n)}(0) = 1$

$$= f(0) + f'(0) \cdot x + f''(0)\frac{x^2}{2!} + \cdots + f^{(n)}(0)\frac{x^n}{n!} + \cdots$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$= \sum \frac{x^k}{k!}$$ (This is also the Maclaurin series of $f(x)$)

Taylor polynomial of order $n$ at $x = 0$ is

$$P_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

Example: Find the Taylor series and Taylor Polynomials generated by $f(x) = \cos x$ at $x = 0$

solution:

$f(x) = \cos x \qquad f'(x) = -\sin x, \quad f''(x) = -\cos x, \quad f'''(x) = \sin x$

$f^{(2n)}(x) = (-1)^n \cos x \qquad f^{(2n+1)}(x) = (-1)^{n+1} \sin x$

at $x = 0$

$f^{(2n)}(0) = (-1)^n \qquad f^{(2n+1)}(0) = 0$

$$f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$= 1 + 0 \cdot x + -\frac{x^2}{2!} + 0 \cdot x^3 + \frac{x^4}{4!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$$

$$= \sum \frac{(-1)^k x^{2k}}{(2k)!}$$

$$P_{2n}(x) = P_{2n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!}.$$