10.9 Convergence of Taylor Serie

Taylors Theorem:

If f and its first n derivatives $f', f'', \ldots, f^{(n)}$ are continuous on the closed interval between a and b and $f^{(n)}$ is differentiable on the open interval between a and b, then there exists a number c between a and b;

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)(b-a)^2}{2!} + \cdots + \frac{f^{(n)}(a)(b-a)^n}{n!} + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

Taylors formula:

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } c \text{ between } a \text{ and } x$$

* So Taylor's Theorem says $\forall x \in I$

$$f(x) = P_n(x) + R_n(x)$$

* R_n depends on the value of $(n+1)$st derivative $f^{(n+1)}$ at a point c, that depends on both a and x

* $\boxed{R_n}$ is called the $\boxed{\text{remainder}}$ of order n or the $\boxed{\text{error term}}$ for the approximation of f by $P_n(x)$ over I.

* $\boxed{\text{If } R_n(x) \longrightarrow 0}$ as $n \to \infty$ for all $x \in I$, we say that the Taylor series generated by f at $x = a$ $\boxed{\text{converges}}$ to f on I, and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

Example 1: show that the Taylor series generated by $f(x) = e^{x}$ at $x=0$, converges to $f(x)$ for every real value of x.

$f(x) = e^{x}$

$\bar{f}(x) = \bar{\bar{f}}(x) = \cdots = f^{(n)}(x) = e^{x}$

so $f(x) = e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + R_n(x)$

$R_n(x) = \frac{e^{c}\, x^{n+1}}{(n+1)!}$ for some c between 0 and x

If $R_n(x) \to 0$ so Taylor series $\to f$

so we have to show that $R_n(x) \to 0$

$R_n(x) = \frac{e^{x}\, x^{n+1}}{(n+1)!}$

The value of $R_n(x)$ depends on e^{x}

e^{x} is increasing function of x

e^{c} lies between e^{0} and e^{x}

① x is negative so $e^{c} < 1$

so $e^{c} < e^{0} = 1$ and $e^{c} \leq 1$ and $R_n(x) = 0$

② $x = 0$

③ If x is positive

$$e^c < e^x$$

$$|R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!}, \quad x \le 0 \quad [e^c < 1]$$

$$|R_n(x)| \le e^x \frac{x^{n+1}}{(n+1)!}, \quad x > 0 \quad [e^c < e^x]$$

In both cases $\lim_{n \to \infty} R_n(x) = 0 \quad \forall x$

So the series converges to e^x $\forall x$.

So

$$e^x = \cancel{\sum \frac{x^k}{k}}$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^k}{k!} + \dots$$

for $x = 1$

$$e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} + R_n(1)$$

$$R_n(1) = e^c \cdot \frac{1}{(n+1)!} < \frac{3}{(n+1)!}$$

Estimating the Remainder.

The Remainder Estimation Theorem:

If there is a positive constant M such that $f^{(n+1)}(t) \le M$ $\forall\, t$ between x and a inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \le \frac{M\,|x-a|^{n+1}}{(n+1)!}$$

If this inequality holds for every n and the other conditions of Taylor Theorem are satisfied by f then the series converges to $f(x)$

Example:
show that the Taylor series for $\sin x$ at $x=0$ converge for all x

solution:

$f(x) = \sin x$, $\bar{f}(x) = \cos x$, $\bar{\bar{f}}(x) = -\sin x$, $\bar{\bar{\bar{f}}}(x) = -\cos x$

$\ldots\ f^{(2k+1)}(x) = (-1)^k \cos x$, $f^{(2k)}(x) = (-1)^k \sin x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^k\, x^{2k+1}}{(2k+1)!} + R_{2k+1}(x)$$

all the derivatives of $\sin x$ have absolute value less than or equal 1 so we can apply the Remainder Estimation Th. RET

$$\left| R_{2k+1}(x) \right| \le \frac{1 \cdot |x|^{2k+2}}{(2k+2)!}$$

$R_{2k+1}(x) \longrightarrow 0$ as $k \to \infty$

So

~~Sin~~ the Maclaurine series for $\sin x$ converges to $\sin x$

$\forall x$

So

$$\sin x = \sum \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots$$

Example:

Show that the Taylor series for $\cos x$ at $x = 0$ converges to $\cos x$ for every value of x

Solution:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + R_{2k}(x)$$

$$R_{2k} = \frac{(-1)^k \cos(x) \cdot |x|^{2k+1}}{(2k+1)!}$$

~~K~~ $M = 1$

so $$\left| R_{2k}(x) \right| \le \frac{1 \cdot |x|^{2k+1}}{(2k+1)!}$$

~~R~~

as $k \to \infty \Rightarrow R_{2k} \to 0$

So $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

Using Taylor Series:

Example: Use known series, find the first few terms of Taylor series for the given function using power series operations

a) $\frac{1}{3}\left(2x + x\cos x\right)$

$$= \frac{2}{3}x + \frac{1}{3}x\cos x$$

$$= \frac{2}{3}x + \frac{1}{3}x\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots\right]$$

$$= \frac{2}{3}x + \frac{1}{3}x - \frac{x^3}{6=3!} + \frac{x^5}{3 \cdot 4!} - \frac{x^7}{3 \cdot 6!} + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{4! \cdot 3} - \frac{x^7}{3 \cdot 6!} + \cdots$$

$$= x - \frac{x^3}{6} + \frac{x^5}{72} - \frac{x^7}{2160} + \cdots$$

(b) $e^x \cos x$

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - \left(\frac{x^2}{2!} + \frac{x^3}{2!} + \frac{x^4}{2!\cdot 2!} + \frac{x^5}{2!\,3!} + \cdots\right)$$

$$+ \left(\frac{x^4}{4!} + \frac{x^5}{4!} + \frac{x^6}{2!\,4!} + \cdots\right) + \cdots$$

$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$$

(c) $$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots$$

$$= 1 - \frac{2^2 x^2}{2!} + \frac{2^4 x^4}{4!} - \frac{2^6 x^6}{6!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} x^{2k}}{(2k)!}$$

Example: For what values of x can we replace $\sin x$ by $x - \frac{x^3}{3!}$ with an error of magnitude no greater than 3×10^{-4}?

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

all terms after $\frac{x^3}{3!}$ are less than or equal $\left|\frac{x^5}{5!}\right|$

so $\frac{|x^5|}{5!} = \frac{|x|^5}{5!} < 3 \times 10^{-4}$

$$|x|^5 < 3 \times 10^{-4} \times 5!$$

$$|x| < \sqrt[5]{3 \times 10^{-4} \times 5!} \simeq 0.514$$

(22) Use power series to find the Taylor series at $x=0$

$$\frac{2}{(1-x)^3} = 2(1-x)^{-3} \xrightarrow{\int} \frac{1}{(1-x)^2} \xrightarrow{\int} \frac{1}{1-x}$$

$$\frac{1}{1-x} = \frac{a}{1-r} \longrightarrow a=1,\ r=x,\ |r|=|x|<1$$

$$-1<x<1$$

$$\frac{1}{1-x} = 1+x+x^2+x^3+x^4+\dots+x^n+\dots = \sum_{n=0}^{\infty} x^n,\quad -1<x<1$$

$$\left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+\dots+nx^{n-1}+\dots = \sum_{n=1}^{\infty} n x^{n-1}$$

$$\left(\frac{1}{(1-x)^2}\right)' = \frac{2}{(1-x)^3} = 2+6x+12x^2+\dots+n(n-1)x^{n-2}+\dots = \sum_{n=2}^{\infty} n(n-1)x^{n-2}$$

$$\boxed{\frac{2}{(1-x)^3} = \sum_{n=0}^{\infty} (n+2)(n+1)x^n}$$