11.2 Calculus with parametric equations

$x = f(t)$

$y = g(t)$

The parametric curve is differentiable at t if f and g are differentiable at t

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Parametric Formula for dy/dx

If all 3 derivatives exist and $\frac{dx}{dt} \neq 0$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} .$$

Now

$$\frac{d^2y}{dx^2} = \frac{d(\bar{y})/dt}{dx/dt}$$

Parametric Formula for $\frac{d^2y}{dx^2}$.

If the equations $x = f(t)$, $y = g(t)$ define y as a twice diff. function of x, then at any Point where

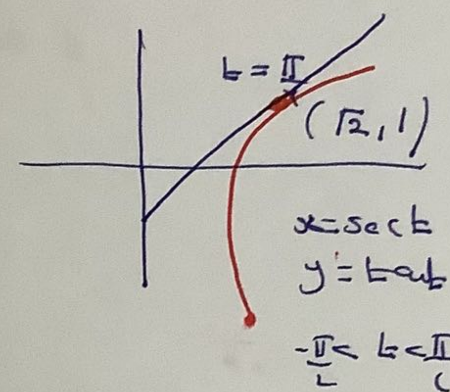
$\frac{dx}{dt} \neq 0$ and $\bar{y} = dy/dx$

$$\frac{d^2y}{dx^2} = \frac{d\bar{y}/dt}{dx/dt}$$

Example: Find the tangent to the curve

$x = \sec t, \quad y = \tan t, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$

at the point $(\sqrt{2}, 1)$ where $t = \frac{\pi}{4}$.



$y - y_1 = m(x - x_1)$

$x_1 = \sqrt{2}, \quad y_1 = 1$

m = unknown

$$m = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sec^2 t}{\sec t \tan t} = \frac{\sec t}{\tan t}$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = \frac{\sec(\frac{\pi}{4})}{\tan\frac{\pi}{4}} = \sqrt{2}$$

$y - 1 = \sqrt{2}(x - \sqrt{2})$

$\boxed{y = \sqrt{2}\, x - 1}$

Example: Find $\frac{d^2y}{dx^2}$ as a function of t

if $x = t - t^2$

$y = t - t^3$

$$\frac{d^2y}{dx^2} = \frac{d\bar{y}/dt}{dx/dt}$$

$$\bar{y} = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{1-2t}$$

$$\boxed{\bar{y} = \frac{1-3t^2}{1-2t}}$$

$$\frac{d\bar{y}}{dt} = \frac{2-6t+6t^2}{(1-2t)^2}$$

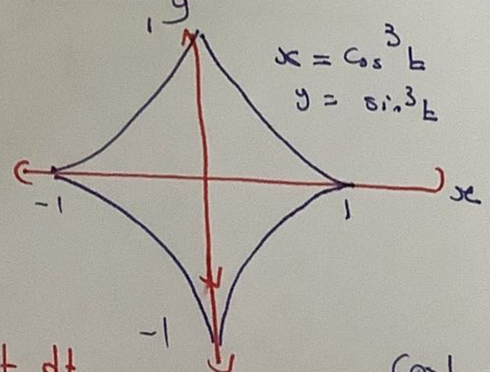
$$\frac{d^2y}{dx^2} = \frac{\frac{2-6t+6t^2}{(1-2t)^2}}{1-2t} = \frac{2-6t+6t^2}{(1-2t)^3}$$

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{d\bar{y}/dt}{dx/dt}, \quad \bar{y} = \frac{dy}{dx}}$$

Example: Find the area enclosed by the astroid

$$x = \cos^3 t, \quad y = \sin^3 t, \quad 0 \le t \le 2\pi$$



$x = \cos^3 t$

$y = \sin^3 t$, $0 \le t \le 2\pi$

$$A = 4\int_0^1 y\, dx$$

$$= 4\int_0^{\frac{\pi}{2}} \sin^3 t \cdot 3\cos^2 t \sin t \, dt$$

$\cos^2 t = \left(\frac{1+\cos 2t}{2}\right)$

$\sin^4 t = \left(\frac{1-\cos 2t}{2}\right)^2$

$$= 12\int_0^{\frac{\pi}{2}} \left(\frac{1-\cos 2t}{2}\right)^2 \left(\frac{1+\cos 2t}{2}\right) dt$$

$$= \frac{3}{2}\int_0^{\frac{\pi}{2}} \left(1+\cos^2 2t - 2\cos 2t\right)\left(1+\cos 2t\right) dt$$

$$= \frac{3}{2}\int 1 - \cos 2t - \cos^2 2t + \cos^3 2t \, dt$$

$$= \frac{3}{2}\left[\int_0^{\frac{\pi}{2}} 1 - \cos 2t \, dt - \int_0^{\frac{\pi}{2}} \cos^2 2t \, dt + \int_0^{\frac{\pi}{2}} \cos^3 2t \, dt\right]$$

$$= \frac{3}{2}\left[t - \frac{\sin 2t}{2} - \frac{1}{2}\left(t + \frac{\sin 2t}{4}\right) + \frac{1}{2}\left(\sin 2t - \frac{1}{3}\sin^3 2t\right)\right]_0^{\frac{\pi}{4}}$$

$$= \frac{3\pi}{8}$$

Length of a parametrically Defined Curve

Def:

Let C be a curve that is defined parametrically by $x = f(t)$ and $y = g(t)$. $a \le t \le b$ where $\bar{f}$ and $\bar{g}$ are continuous on $[a, b]$ and not simultaneously zero and C is traversed exactly once as t increases from $t = a$ to $t = b$, then the length of the curve C is

$$L = \int_a^b \sqrt{[\bar{f}(t)]^2 + [\bar{g}(t)]^2} \, dt$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

Ex] Use the def. to find the length of the circle of radius defined parametrically by

$$x = r\cos t \quad , \quad y = r\sin t \quad , \quad 0 \le t \le 2\pi$$

Solution

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{(r\sin t)^2 + (r\cos t)^2}\, dt$$

$$= \int_0^{2\pi} r\, dt = rt = r(2\pi - 0) = 2\pi r$$

Ex] Find the length of the astroid

$$x = \cos^3 t \qquad y = \sin^3 t \qquad 0 \le t \le 2\pi$$

$$\frac{dx}{dt} = -3\cos^2 t \sin t \quad , \quad \left(\frac{dx}{dt}\right)^2 = 9\cos^4 t \sin^2 t$$

$$\frac{dy}{dt} = 3\sin^2 t \cos t \quad , \quad \left(\frac{dy}{dt}\right)^2 = 9\sin^4 t \cos^2 t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9\sin^2 t \cos^2 t \left[\cos^2 t + \sin^2 t\right]$$

$$= 9\sin^2 t \cos^2 t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 3\,|\sin t \cos t| = 3\sin t \cos t \quad , \quad 0 \le t \le \frac{\pi}{2}$$

d

$$L = 4\int_0^{\frac{\pi}{2}} 3 \sin t \cos t \, dt$$

$$= -4 \cdot \frac{3}{2} \, \frac{\cos 2t}{2}\Bigg|_0^{\frac{\pi}{2}} \doteq 6$$