Test for convergence and Divergence.

Example: ① Does the integral $\int_1^\infty e^{-x^2}\,dx$ converge?

we can't evaluate $\int e^{-x^2}\,dx$ since it's nonelementary

Note that $x^2 > x$ for $x \geq 1$

$-x^2 < -x$ for $x \geq 1$

$e^{-x^2} < e^{-x}$ for $x \geq 1$

So $\int_1^\infty e^{-x^2}\,dx < \int_1^\infty e^{-x}\,dx$

$$\int_1^\infty e^{-x}\,dx = \lim_{b\to\infty} \int_1^b e^{-x}\,dx = \lim_{b\to\infty} -e^{-x}\Big|_1^b = \lim_{b\to\infty} -e^{-b} + e^{-1}$$

$$= 0 + e^{-1} = \frac{1}{e}$$

So we know that $\int_1^\infty e^{-x^2}\,dx$ converges to some definite finite value.

Theorem: Direct Comparison Test

Let f and g be continuous on $[a,\infty)$ with $0 \le f(x) \le g(x)$ for all $x \geq a$, then

① $\int_a^\infty f(x)\,dx$ converges if $\int_a^\infty g(x)\,dx$ converges

② $\int_a^\infty g(x)\,dx$ ~~diverges~~ diverges if $\int_a^\infty f(x)\,dx$ diverges

Example:

ⓐ $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$

$\frac{0}{x^2} \le \frac{\sin^2 x}{x^2} \le \frac{1}{x^2}$ on $[1,\infty)$

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{a\to\infty} \int_1^a x^{-2}\,dx = \lim_{a\to\infty} -x^{-1}\Big|_1^a$$

$$= \lim_{a\to\infty} -a^{-1}+1 = \lim_{a\to\infty} -\frac{1}{a}+1 = 1$$

so $\int_1^\infty \frac{1}{x^2}\,dx$ converges

so $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ converges using D.C.T. with $\int_1^\infty \frac{1}{x^2}\,dx$

b) $$\int_1^{\infty} \frac{1}{\sqrt{x^2-0.1}}\,dx$$

$$x^2-0.1 < x^2$$

$$\sqrt{x^2-0.1} < \sqrt{x^2} = |x| = x$$

$$\frac{1}{\sqrt{x^2-0.1}} > \frac{1}{x}$$

$\int_1^{\infty} \frac{1}{x}\,dx$ diverges using The integral $\int_1^{\infty} \frac{dx}{x^p}$

So $$\int_1^{\infty} \frac{1}{\sqrt{x^2-1}}\,dx > \int_1^{\infty} \frac{1}{x}\,dx$$

$\int_1^{\infty} \frac{1}{\sqrt{x^2-1}}\,dx$ diverges using D.C.T

Limit Comparison Test

If the positive functions f and g are continuous on $[a, \infty)$ and if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L, \quad 0 < L < \infty$$

then.

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

both diverge or both converge.

Example: show that $\int_1^\infty \frac{dx}{1+x^2}$ converges by comparison with $\int_1^\infty \frac{1}{x^2}\,dx$

$$\int_1^\infty \frac{1}{x^2}\,dx \quad \text{converges to } \frac{1}{2-1} = 1$$

$$\lim_{x \to \infty} \frac{\frac{1}{1+x^2}}{\frac{1}{x^2}} = \lim_{x \to \infty} \frac{x^2}{x^2+1} = 1$$

so also $\int_1^\infty \frac{dx}{x^2+1}$ converges using L.C.T with $\int_1^\infty \frac{1}{x^2}$

Example: Investigate the Convergence of

$$\int_1^\infty \frac{1-e^{-x}}{x}\, dx$$

$$\frac{1-e^{-x}}{x} , \frac{1}{x}$$

$\int_1^\infty \frac{1}{x} dx$ diverge.

$$\lim_{x\to\infty} \frac{\frac{1-e^{-x}}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x(1-e^{-x})}{x} = \lim_{x\to\infty} 1-e^{-x} = 1$$

so $\int_1^\infty \frac{1-e^{-x}}{x} dx$ diverges using L.C.T with $\int_1^\infty \frac{1}{x} dx$