8.7 Improper Integrals

Until now, all Integrals were discussed have two Properties. "Definite Integral"

1. Domain of the integration $[a,b]$ is finite
2. Range of the integrand be finite on this Domain

The integral for the area under the curve $y = \frac{\ln x}{x^2}$ from $x = 1$ to $x = \infty$ is an example for which the domain is infinite

and the integral for the area under the curve of $y = \frac{1}{\sqrt{x}}$ between $x = 0$ and $x = 1$ is an example of integrands of infinite range

In both cases, the integrals are said to be improper and are collected as Limits

Definition:

Integrals with infinite limits of integration are improper integrals of Type I

1. If $f(x)$ is continuous on $[a,\infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$

2. If f is continuous on $(-\infty,b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$

3. If $f(x)$ is continuous on $(-\infty,\infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

where c is any real number

In each case, if the limit is finite we say that the improper integral $\boxed{\text{converges.}}$

If the limit fails to exist, the improper integral $\boxed{\text{diverges}}$

Example: Is the area under $y = \frac{\ln x}{x^2}$ from $x = 1$ to $x = \infty$ finite? If so, what is its value?

Solution:

$$\int_1^\infty \frac{\ln x}{x^2}\,dx$$

$$\int \frac{\ln x}{x^2}\,dx$$

$\ln x = u$ $\qquad x^{-2}\,dx = dv$

$\frac{dx}{x} = du$ $\qquad -x^{-1} = v$

$-\frac{1}{x} = v$

$$\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx$$

$$= \frac{-\ln x}{x} + -\frac{1}{x}$$

Returning to our Integral

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \left(-\frac{\ln x}{x} - \frac{1}{x}\right)\Big|_1^b$$

$$= \lim_{b\to\infty} \left(\frac{-\ln b}{b} - \frac{1}{b}\right) - (0 - 1)$$

$$= \lim_{b\to\infty} \frac{-\ln b}{b} - \frac{1}{b} + 1$$

$$= \lim_{b\to\infty}\left(-\frac{\ln b}{b}\right) + 1$$

$$= \lim_{b\to\infty}\left(\frac{-\frac{1}{b}}{1}\right) + 1$$

$$= 0+1=1$$

the improper integral converges.

area $=1$

Example: $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{1+x^2}$

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \int_{-\infty}^{0}\frac{dx}{1+x^2} + \int_{0}^{\infty}\frac{dx}{1+x^2}$$

$$\int\frac{dx}{1+x^2} = \tan^{-1}x$$

$$\int_{-\infty}^{0}\frac{dx}{1+x^2} = \lim_{a\to-\infty}\int_{a}^{0}\frac{dx}{1+x^2} = \lim_{a\to-\infty}\tan^{-1}x\Big|_{a}^{0}$$

$$= \lim_{a\to-\infty}\tan^{-1}0 - \tan^{-1}a$$

$$= \cancel{\lim}\quad 0 - -\frac{\pi}{2} = \frac{\pi}{2}$$

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_0^b \frac{dx}{1+x^2}$$

$$= \lim_{b\to\infty} \tan^{-1} x \Big|_0^b$$

$$= \lim_{b\to\infty} \tan^{-1} b - \tan^{-1} 0$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

So $\displaystyle\int_{-\infty}^\infty \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

The Integral $\displaystyle\int_1^\infty \frac{dx}{x^p}$

Example 3:
For what values of p, does the integral $\displaystyle\int_1^\infty \frac{dx}{x^p}$ converge? when the integral does converge, what is its value?

$$\int_1^b \frac{dx}{x^p} = \int_1^b x^{-p}\,dx = \frac{x^{-p+1}}{-p+1}\Big|_1^b, \quad p \neq 1$$

$$= \frac{b^{-p+1} - 1}{-p+1}$$

$$\int_1^\infty \frac{dx}{x^p} = \lim_{a\to\infty} \int_1^a \frac{dx}{x^p}$$

$$= \lim_{a\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right|_1^a \quad , p \neq 1$$

$$= \lim_{a\to\infty} \frac{a^{-p+1} - 1}{-p+1}$$

$$= \lim_{a\to\infty} \frac{1}{1-p}\left(\frac{1}{a^{1-p}} - 1\right) = \begin{cases} \frac{1}{p-1} , & p > 1 \\ \infty , & p < 1 \end{cases}$$

Recall :

$$\lim_{y\to\infty} x^y = \begin{cases} \infty , & |x|>1 \\ 0 , & |x|<1 \end{cases}$$

So if $p>1$, the integral converges to $\frac{1}{p-1}$

if $p<1$, the integral diverges

if $p=1$

$$\int_1^\infty \frac{dx}{x} = \lim_{a\to\infty} \ln|x| \Big|_1^a = \lim_{a\to\infty} \ln a - \ln 1$$

$$= \infty$$

Diverge

Integrand with vertical Asymptotes

Definition

Integrals of functions that become infinite at a point within the interval of integration are improper integral of Type II

1. If $f(x)$ is continuous on $(a,b]$ and discontinuous ~~on [a,b] or~~ at a, then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx$$

2. If $f(x)$ is continuous on $[a,b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$$

3. If $f(x)$ is discontinuous at c, where $a<c<b$ and continuous on $[a,c) \cup (c,b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Example: Investigate the convergence of

$$\int_0^1 \frac{1}{1-x}\,dx$$

$1-x=0$

$x=1$

$\frac{1}{1-x}$ is discontinuous at $x=1$

$$\lim_{b\to 1^-}\int_0^b \frac{1}{1-x}\,dx = \lim_{b\to 1^-} -\ln|1-x|\Big|_0^b$$

$$=\lim_{b\to 1^-}\left(-\ln|1-b| + \ln 1\right)$$

$$= \infty + 0 = \infty$$

So the integral diverges

Example: Evaluate.

$$\int_0^3 \frac{dx}{(x-1)^{2/3}}$$

the integrand $\frac{1}{(x-1)^{\frac{2}{3}}}$ is discontinuous at $x=1$

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{\frac{2}{3}}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}$$

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-} \int_0^b \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-} 3(x-1)^{\frac{1}{3}}\Big|_0^b$$

$$= \lim_{b\to 1^-} 3(b-1)^{\frac{1}{3}} - 3(-1)^{\frac{1}{3}}$$

$$= 3$$

$$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{a\to 1^+} \int_a^3 \frac{dx}{(x-1)^{2/3}} = \lim_{a\to 1^+} 3(x-1)^{\frac{1}{3}}\Big|_a^3$$

$$= \lim_{a\to 1^+} 3\cdot 2^{\frac{1}{3}} - 3(a-1)^{\frac{1}{3}}$$

$$= 3\ 2^{1/3}$$

So $\displaystyle\int_0^3 \frac{dx}{(x-1)^{2/3}} = 3 + 3\cdot\sqrt[3]{2}$

Converges