10.1 Sequences.

6. $a_n = \frac{2^n - 1}{2^n}$. Find the values of a_1, a_2, a_3 & a_4.

$a_1 = \frac{2^1 - 1}{2^1} = \frac{1}{2}$, $a_2 = \frac{3}{4}$, $a_3 = \frac{7}{8}$, $a_4 = \frac{15}{16}$, ...

---

10. $a_1 = -2$, $a_{n+1} = \frac{n a_n}{n+1}$, write the first ten terms of the sequence.

$a_1 = -2$, $\boxed{n=1}$ $a_{1+1} = \frac{1(a_1)}{1+1}$

$a_2 = \frac{-2}{2} = -1$

$\boxed{n=2}$ $a_{2+1} = \frac{2\, a_2}{2+1} = \frac{2(-1)}{3}$

$a_3 = -\frac{2}{3}$

$\boxed{n=3}$ $\frac{3\, a_3}{3+1} = a_{3+1}$

$a_4 = \frac{3\left(-\frac{2}{3}\right)}{4} = -\frac{1}{2}$

$\boxed{n=4}$ $a_{4+1} = \frac{4\, a_4}{4+1} = \frac{4\left(-\frac{1}{2}\right)}{5} = -\frac{2}{5}$

⋮

---

22. 2, 6, 10, 14, 18, ... Find a formula for nth term of the sequence.

Note arithmetic sequence

$a_n = a_1 + (n-1) d$

d common difference

$a_1 = 2$

$\left.\begin{array}{l} 6-2=4 \\ 10-6=4 \\ 14-10=4 \end{array}\right\} \Rightarrow d=4.$

So, $a_n = 2 + (n-1)4$

$a_n = 2 + 4n - 4$

$a_n = -2 + 4n$. $n = 1, 2, \ldots$

26 $0, 1, 1, 2, 2, 3, 3, \ldots$, find a formula for nth term of the sequence

$\left[\frac{1}{2}\right] = 0$

$\left[\frac{2}{2}\right] = [1] = 1$

$\left[\frac{3}{2}\right] = 1$

$\left[\frac{4}{2}\right] = [2] = 2$

$\left[\frac{5}{2}\right] = 2$

$\vdots$

So $a_n = \left[\frac{n}{2}\right]$. $\quad n = 1, 2, \ldots$

---

31) $a_n = \dfrac{1 - 5n^4}{n^4 + 8n^3}$, Is the sequence converges or diverges, find its limit?

$$a_n = \frac{\frac{1}{n^4} - 5}{1 + \frac{8}{n}}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\frac{1}{n^4} - 5}{1 + \frac{8}{n}}$$

$$= \frac{-5}{1} = -5 \quad \text{Converges.}$$

---

35) $a_n = 1 + (-1)^n$

$a_n = 0, 2, 0, 2, \ldots$

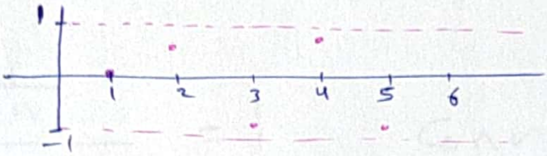
$\lim_{n\to\infty} a_n$ does not exist.

diverges.

(36) $a_n = (-1)^n \left(1 - \frac{1}{n}\right)$.

$a_n = 0, \frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \dots$

$\lim_{n\to\infty} a_n$ does not exist. $\Rightarrow$ diverges



(41) $a_n = \sqrt{\frac{2n}{n-1}}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{\frac{2n}{n-1}} = \sqrt{\lim_{n\to\infty} \frac{2n}{n-1}} = \sqrt{2}$. Converges.

(44) $a_n = n\pi \cos(n\pi)$

$a_n = -\pi, 2\pi, -3\pi, 4\pi, \dots$, $\lim_{n\to\infty} n\pi \cos(n\pi) = \lim_{n\to\infty} n\pi (-1)^n$

$\lim_{n\to\infty} a_n$ does not exist $\Rightarrow$ diverges.

(48) $a_n = \frac{3^n}{n^3}$

$\lim_{n\to\infty} \frac{3^n}{n^3}$ $\frac{\infty}{\infty}$

$= \lim_{n\to\infty} \frac{\ln 3 \cdot 3^n}{3n^2}$

$= \lim_{n\to\infty} \frac{(\ln 3)^2\, 3^n}{6n}$

$= \lim_{n\to\infty} \frac{(\ln 3)^3\, 3^n}{6} = \infty$

diverges.

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50) $a_n = \dfrac{\ln n}{\ln 2n}$

$$\lim_{n\to\infty} \frac{\ln n}{\ln 2n} \quad \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{2}{2n}} = 1 \quad \text{Converges.}$$

---

54) $a_n = \left(1 - \dfrac{1}{n}\right)^n$

$a_n = \left(1 + \dfrac{(-1)}{n}\right)^n$ $\qquad \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$

$$\lim_{n\to\infty} \left(1 + \frac{(-1)}{n}\right)^n = e^{-1} = \frac{1}{e} \quad \text{(Converges).}$$

---

70) $\lim_{n\to\infty} \left(\dfrac{n}{n+1}\right)^n$

$$= \lim_{n\to\infty} e^{n \ln\left(\frac{n}{n+1}\right)} = \lim_{n\to\infty} e^{n\left(\ln n - \ln(n+1)\right)}$$

$$= \lim_{n\to\infty} e^{\left[\frac{\ln n - \ln(n+1)}{\frac{1}{n}}\right]}$$

$$= \lim_{n\to\infty} e^{\left[\frac{\frac{1}{n} - \frac{1}{n+1}}{-\frac{1}{n^2}}\right]} = \lim_{n\to\infty} e^{\left[-n^2\left(\frac{n+1-n}{n(n+1)}\right)\right]}$$

$$= \lim_{n\to\infty} e^{\frac{-n^2}{n(n+1)}}$$

$$= e^{-1} = \frac{1}{e} \quad \text{Converges.}$$

58) $a_n = (n+4)^{\frac{1}{n+4}}$

let $u = n+4$.

$n \to \infty \Rightarrow u \to \infty$.

$\lim_{u \to \infty} u^{\frac{1}{u}} = 1$

So $\lim_{n \to \infty} a_n = \lim_{n \to \infty} (n+4)^{\frac{1}{n+4}} = 1$

Converges.

---

60) $a_n = \ln n - \ln(n+1)$.

$a_n = \ln\left(\frac{n}{n+1}\right)$

$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \ln\left(\frac{n}{n+1}\right)$

$= \ln\left(\lim_{n \to \infty}\left(\frac{n}{n+1}\right)\right)$

$= \ln(1) = 0$. Converges.

---

63) $a_n = \frac{n!}{n^n}$

$a_n = \frac{n}{n} \frac{n-1}{n} \frac{n-2}{n} \cdot \dots \frac{2}{n} \cdot \frac{1}{n}$.

$0 \leq a_n \leq \frac{1}{n}$.

$\underbrace{\lim_{n \to \infty} 0}_{0} \leq \lim_{n \to \infty} a_n \leq \underbrace{\lim_{n \to \infty} \frac{1}{n}}_{0}$

by Sandwich Theorem $\lim_{n \to \infty} a_n = 0$.

Converges.

72) $a_n = \left(1 - \frac{1}{n^2}\right)^n$

$$a_n = \left[\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right)\right]^n$$

$$= \left(1+\frac{-1}{n}\right)^n \left(1+\frac{1}{n}\right)^n$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{-1}{n}\right)^n \cdot \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$$= e^{-1} \cdot e^{1}$$

$$= e^{0}$$

$$= \boxed{1}$$

remember: $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$ x is fixed

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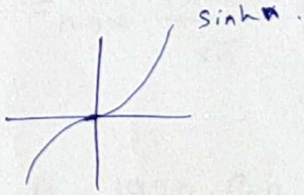
76) $a_n = \sinh(\ln n)$



$$\sinh n = \frac{e^{n} - e^{-n}}{2}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sinh(\ln n)$$

$$= \lim_{n\to\infty} \frac{e^{\ln n} - e^{-\ln n}}{2}$$

$$= \lim_{n\to\infty} \frac{n - \frac{1}{n}}{2}$$

$$= \infty \quad \text{diverges.}$$

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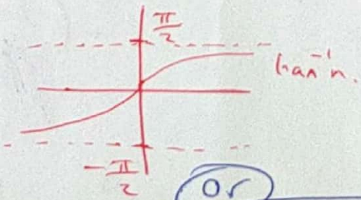
82) $a_n = \frac{\tan^{-1} n}{\sqrt{n}}$



$$-\frac{\pi}{2} < \tan^{-1} n < \frac{\pi}{2}$$

$$-\frac{\pi}{2\sqrt{n}} < \frac{\tan^{-1} n}{\sqrt{n}} < \frac{\pi}{2\sqrt{n}}$$

$\lim_{n\to\infty} -\frac{\pi}{2\sqrt{n}} = 0$

& $\lim_{n\to\infty} \frac{\pi}{2\sqrt{n}} = 0$

So, by sandwich Theorem $\lim_{n\to\infty} \frac{\tan^{-1} n}{\sqrt{n}} = 0.$

or

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \cdot \tan^{-1} n$$

$$= 0 \cdot \frac{\pi}{2}$$

$$= 0.$$

Converges.

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86) $a_n = \frac{(\ln n)^5}{\sqrt{n}}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(\ln n)^5}{\sqrt{n}} \quad \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{\frac{5(\ln n)^4}{n}}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{10(\ln n)^4}{\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{\frac{40(\ln n)^3}{n}}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{80(\ln n)^3}{\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{\frac{240(\ln n)^2}{n}}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{480(\ln n)^2}{\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{\frac{960 \ln n}{n}}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{1920 \ln n}{\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{\frac{1920}{n}}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{3840}{\sqrt{n}}$$

$$= \boxed{0}$$ Converges.

---

92) $a_{n+1} = \frac{a_n + 6}{a_n + 2}$, $a_1 = -1$. (assume the sequence is converges & find the limit).

Since a_n converges $\Rightarrow \lim_{n\to\infty} a_n = L$

$$\& \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{a_n + 6}{a_n + 2}$$

$$L = \frac{L+6}{L+2}$$

$$L(L+2) = L+6$$

$$L^2 + 2L - L - 6 = 0$$

$L^2 + 6 - 6 = 0.$

$(L+3)(L-2) = 0$

$L = -3$ or $L = 2.$

Since $a_n > 0$ for $n \geq 2$

So, $\boxed{L = 2}$

III) Determine if the following sequence is monotonic & if it is bounded.

$$a_n = \frac{3n+1}{n+1} \qquad \left[2, \frac{7}{3}, \frac{10}{4}, \frac{13}{5}, \dots\right]$$

$a_{n+1} > a_n.$

$$\frac{3(n+1)+1}{(n+1)+1} > \frac{3n+1}{n+1}$$

$$\frac{3n+4}{n+2} > \frac{3n+1}{n+1}$$

$3n^2+7n+4 > 3n^2+7n+2.$

$4 > 2$ ✓

$\therefore$ The sequence is nondecreasing (monotonic).

Now, $\frac{3n+1}{n+1} < 3.$

$3n+1 < 3n+3$

$1 < 3$

$\therefore$ The sequence is bounded above by 3 & bounded below by 2.

$\Rightarrow a_n$ is bounded.

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10.2 Infinite Series.

8 Find a formula for the nth partial sum & use it to find the series' sum:-

$$\frac{5}{1}+\frac{5}{2(3)}+\frac{5}{3(4)}+\dots+\frac{5}{n(n+1)}+\dots$$

$$a_n=\frac{5}{n(n+1)}=\frac{A}{n}+\frac{B}{n+1}$$

* $A=5$ & $B=-5$

So, $\frac{5}{n(n+1)}=\frac{5}{n}-\frac{5}{n+1}$

$$S_n=\sum_{n=1}^{\infty}\left(\frac{5}{n}-\frac{5}{n+1}\right)$$ Telescoping..

$$S_n=\left(\frac{5}{1}-\cancel{\frac{5}{2}}\right)+\left(\cancel{\frac{5}{2}}-\cancel{\frac{5}{3}}\right)+\left(\cancel{\frac{5}{3}}-\cancel{\frac{5}{4}}\right)+\dots+\cancel{\frac{5}{n}}-\frac{5}{n+1}$$

$$\boxed{S_n=5-\frac{5}{n+1}}$$ The nth partial sum.

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left(5-\frac{5}{n+1}\right)=5.$$

So, $\sum_{n=1}^{\infty}\frac{5}{n(n+1)}=5$ "That is Converges to 5"

$\boxed{14}$ $\sum_{n=0}^{\infty} \frac{2^{n+1}}{5^n}$

$$\sum_{n=0}^{\infty} \frac{2^{n+1}}{5^n} = \frac{2}{1} + \frac{4}{5} + \frac{8}{25} + \frac{16}{125} + \dots$$

its 'geometric series' with $a = 2$ & $r = \frac{2}{5} < 1$

So, $\sum_{n=0}^{\infty} \frac{2^{n+1}}{5^n}$ converges to $\frac{a}{1-r} = \frac{2}{1-\frac{2}{5}} = \frac{10}{3}$

$$\sum_{n=0}^{\infty} \frac{2^{n+1}}{5^n} = \frac{10}{3}$$

$\boxed{18}$ $\left(-\frac{2}{3}\right)^2 + \left(\frac{-2}{3}\right)^3 + \left(\frac{-2}{3}\right)^4 + \dots$

$$= \sum_{n=2}^{\infty} \left(-\frac{2}{3}\right)^n$$

its a geometric series with $\boxed{a} = \left(-\frac{2}{3}\right)^2 = \frac{4}{9}$

& $\boxed{r} = -\frac{2}{3}$.

$|r| < 1$ $\left(\left|-\frac{2}{3}\right| < 1\right)$, so the series converges

$$\sum_{n=2}^{\infty} \left(\frac{-2}{3}\right)^n = \frac{a}{1-r} = \frac{\frac{4}{9}}{1-\left(-\frac{2}{3}\right)} = \frac{\frac{4}{9}}{1+\frac{2}{3}} = \frac{\frac{4}{9}}{\frac{5}{3}}$$

$$= \frac{4}{9} \cdot \frac{3}{5} = \boxed{\frac{4}{15}}$$

$\boxed{24}$ Express the following as the ratio of two integers.

$$1.\overline{414} = 1 + 0.414 + 0.000414 + \dots$$

$$= 1 + \frac{414}{1000} + \frac{414}{1000000} + \dots$$

☆ Geometric series, with $a = \frac{414}{1000}$ & $r = \frac{1}{1000}$

Converges to $\frac{a}{1-r} = \frac{\frac{414}{1000}}{1-\frac{1}{1000}} = \frac{414}{999}$.

→ So, $1.\overline{414} = 1 + \frac{414}{999} = \frac{1413}{999}$.

$\boxed{32}$ $\sum_{n=0}^{\infty} \frac{e^n}{e^n+n}$

☆ → by nth term test:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^n}{e^n+n} = \lim_{n\to\infty} \frac{e^n}{e^n+1}$$

$$= \lim_{n\to\infty} \frac{e^n}{e^n} = \lim_{n\to\infty} 1 = 1 \neq 0.$$

$\therefore \sum_{n=0}^{\infty} \frac{e^n}{e^n+n}$ diverges by nth term test.

$\boxed{38}$ $\sum_{n=1}^{\infty}(\tan n - \tan(n-1))$ A telescoping series.

$S_n = (\tan 1 - \tan 0) + (\cancel{\tan 2} - \cancel{\tan 1}) + \cdots + (\tan n - \cancel{\tan(n-1)})$

$= -\tan 0 + \tan n.$

$= \tan n.$

by nth partial sum:-

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \tan n = DNE.$

$\therefore \sum_{n=1}^{\infty}(\tan n - \tan(n-1))$ diverges by n^{th} partial sum.

$\boxed{44}$ $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$

$$\frac{2n+1}{n^2(n+1)^2} = \frac{A}{n} + \frac{B}{n^2} + \frac{C}{n+1} + \frac{D}{(n+1)^2}$$

$A=0$, $B=1$, $C=0$, $D=-1$.

So, $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = \sum_{n=1}^{\infty}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right)$

$S_n = \left(1 - \cancel{\frac{1}{2^2}}\right) + \left(\cancel{\frac{1}{2^2}} - \cancel{\frac{1}{3^2}}\right) + \cdots + \left(\cancel{\frac{1}{n^2}} - \frac{1}{(n+1)^2}\right)$

$S_n = 1 - \frac{1}{(n+1)^2}$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{(n+1)^2}\right)$$

$$= 1 - 0 = 1$$

So, the series converges to $\underline{1}$.

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54 $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{5^n}$

$$= \frac{\cos(0)}{1} + \frac{\cos(\pi)}{5} + \frac{\cos(2\pi)}{5^2} + \dots$$

$$= 1 - \frac{1}{5} + \frac{1}{5^2} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{5^n} = \sum_{n=0}^{\infty}\left(\frac{-1}{5}\right)^n.$$

Geometric series with $a = 1$ & $r = -\frac{1}{5}$.

$|r| = \left|-\frac{1}{5}\right| < 1$, so its converges to $\frac{a}{1-r}$

$$\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{5^n} = \frac{1}{1 - -\frac{1}{5}} = \frac{1}{\frac{6}{5}} = \boxed{\frac{5}{6}}$$

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$\boxed{62}$ $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

by nth term test

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^n}{n!} = \lim_{n\to\infty} \frac{n \cdot n \cdot \dots \cdot n}{n \cdot (n-1), \dots \cdot 1}$$

$$= \lim_{n\to\infty} \frac{n}{n} \cdot \lim_{n\to\infty} \frac{n}{n-1} \cdot \dots \cdot \lim_{n\to\infty} \frac{n}{1}$$

$$= (1) \quad (1) \dots \infty$$

$$= \infty \neq 0.$$

$\therefore \sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges by nth term test.

$\boxed{63}$ $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n}$

$$= \sum_{n=1}^{\infty} \left(\frac{2}{4}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n.$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n.$$

$$= \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right) + \left(\frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \dots\right).$$

geometric seriess

$a = \frac{1}{2}$ $\qquad$ $a = \frac{3}{4}$.

$r = \frac{1}{2} < 1$ $\qquad$ $r = \frac{3}{4} < 1$

$$= \frac{\frac{1}{2}}{1-\frac{1}{2}} + \frac{\frac{3}{4}}{1-\frac{3}{4}}$$

$$= 1 + 3 = \boxed{4}$$

$\boxed{78}$ Find the values of x for which the series converges & find the sum.

$$\sum_{n=0}^{\infty} (\ln x)^n$$

$$= 1 + \ln x + (\ln x)^2 + \dots$$

geometric series with $\boxed{a = 1}$ & $\boxed{r = \ln x}$

→ To be converges:

$$|r| < 1$$

$$\rightarrow -1 < r < 1$$

$$-1 < \ln x < 1$$

$$e^{-1} < x < e^{1}$$

So, if $\frac{1}{e} < x < e$, then the series $\sum_{n=0}^{\infty} (\ln x)^n$ converges to $\frac{a}{1-r} = \frac{1}{1-\ln x}$

90 Find the values of b for which

$1 + e^{b} + e^{2b} + e^{3b} + \cdots = 9.$

$$\sum_{n=0}^{\infty} (e^{b})^{n} = 9.$$

geometric series with $a = 1$ & $r = e^{b}$.

$$\sum_{n=0}^{\infty} (e^{b})^{n} = \frac{1}{1 - e^{b}} = 9.$$

$$1 = 9 - 9e^{b}.$$

$$9e^{b} = 8.$$

$$e^{b} = \frac{8}{9}$$

$$b = \ln\left(\frac{8}{9}\right).$$