10.3 The Integral Test.

$\boxed{6}$ $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$, use the integral test.

let $f(x) = \frac{1}{x(\ln x)^2}$, $f(x)$ is +ve , continuous & decreasing $\forall x \geq 2$

* $f(x)$ is decreasing since:

$$f'(x) = \frac{-\left[2\frac{x \ln x}{x} + (\ln x)^2\right]}{x^2(\ln x)^4} = \frac{-2\ln x}{x^2(\ln x)^4} - \frac{(\ln x)^2}{x^2(\ln x)^4}$$

$$= \frac{-2}{x^2(\ln x)^3} - \frac{1}{x^2(\ln x)^2}$$

f' sign 2 - - - - →

Now, To find $\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx$

let $u = \ln x \Rightarrow du = \frac{dx}{x}$

when $x = 2 \Rightarrow u = \ln 2$.
$x = \infty \Rightarrow u = \infty$

$$\Rightarrow \int_{\ln 2}^{\infty} \frac{1}{u^2}\,du = \lim_{b\to\infty} \int_{\ln 2}^{b} u^{-2}\,du$$

$$= \lim_{b\to\infty} \left(-\frac{1}{u} \Big|_{\ln 2}^{b} \right) = \lim_{b\to\infty} \left(-\frac{1}{b} + \frac{1}{\ln 2} \right)$$

$$= \frac{1}{\ln 2}$$ converges.

$\therefore \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges by integral test.

$\boxed{13}$ $\sum_{n=1}^{\infty} \frac{n}{n+1}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$.

by n-th term test $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges.

$\boxed{20}$ $\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$

*remember: $\ln x > 0 \quad \forall x > 1$.

Try to use the integral test.

let $f(x) = \frac{\ln x}{\sqrt{x}}$, $f(x)$ is positive, continuous & decreasing $\forall x \geq 8$.

Since $f'(x) = \frac{\frac{\sqrt{x}}{x} - \frac{\ln x}{2\sqrt{x}}}{(\sqrt{x})^2} = \frac{2 - \ln x}{2x^{3/2}}$

$2 - \ln x < 0$

$\ln x > 2$.

$x > e^2$

f' sign: $+$ on $(2, e^2)$, 0 at e^2, $-$ after

$e^2 \simeq 7.4$.

$$\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}} = \sum_{n=2}^{7} \frac{\ln n}{\sqrt{n}} + \sum_{n=8}^{\infty} \frac{\ln n}{\sqrt{n}}.$$

So, we can apply the integral test for $n \geq 8$.

Now, $\int_8^{\infty} \frac{\ln x}{\sqrt{x}} dx$.

let $u = \ln x \Rightarrow du = \frac{dx}{x}$

$e^u = x \Rightarrow e^u du = dx$.

$e^{u/2} = \sqrt{x}$

when $x = 8 \Rightarrow u = \ln 8$.

$x = \infty \Rightarrow u = \infty$.

$$= \int_{\ln 8}^{\infty} \frac{u}{e^{u/2}} \cdot e^u du. = \int_{\ln 8}^{\infty} u e^{u/2} du.$$

$$= \lim_{b\to\infty} \int_{\ln 8}^{b} u e^{u/2} du.$$

deriv	Integral
u	$e^{u/2}$
1	$+\ 2e^{u/2}$
0	$-\ 4e^{u/2}$

$$= \lim_{b\to\infty} \left(\left(2ue^{u/2} - 4e^{u/2}\right)\Big|_{\ln 8}^{b} \right)$$

$$= \lim_{b\to\infty} \left(2e^{u/2}(u-2) \Big|_{\ln 8}^{b} \right).$$

$$= \lim_{b\to\infty} \left(2e^{b/2}(b-2) - 2e^{\ln 8/2}(\ln 8 - 2) \right).$$

$= \infty$ diverges.

by the integral test $\sum_{n=8}^{\infty} \frac{\ln}{\sqrt{n}}$ diverges.

So, $\sum_{n=2}^{\infty} \frac{\ln 2}{\sqrt{n}}$ diverges.

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$\boxed{22}$ $\sum_{n=1}^{\infty} \frac{5^n}{4^n+3}$

Use n-th term test:

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5^n}{4^n+3} \quad \frac{\infty}{\infty}$

$= \lim_{n\to\infty} \frac{\ln 5 \; 5^n}{\ln 4 \; 4^n + 0} = \frac{\ln 5}{\ln 4} \lim_{n\to\infty} \left(\frac{5}{4}\right)^n \qquad \frac{5}{4} > 1$

$= \infty \neq 0.$

$\Rightarrow$ by n-th term test the series diverges.

$\boxed{28}$ $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$

by using n-th term test :

$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e^1 = e \neq 0$.

So the series diverges.

$\boxed{32}$ $\sum_{n=1}^{\infty} \frac{1}{n(1+\ln^2 n)}$

let $f(x) = \frac{1}{x(1+\ln^2 x)}$ [positive, continuous & decreasing for $x \geq 1$]

$f'(x) = \frac{-\left(x \cdot 2\ln x \cdot \frac{1}{x} + (1+\ln^2 x)\right)}{x^2(1+\ln^2 x)^2} = \frac{-2\ln x - (1+\ln^2 x)}{x^2(1+\ln^2 x)^2}$

$= \frac{-2\ln x}{x^2(1+\ln^2 x)^2} - \frac{1}{x^2(1+\ln^2 x)}$

Now, $\int_1^{\infty} \frac{1}{x(1+\ln^2 x)} dx$

let $u = \ln x \Rightarrow du = \frac{dx}{x}$

when $x = 1 \Rightarrow u = 0$.

$x = \infty \Rightarrow u = \infty$.

$= \int_0^{\infty} \frac{du}{1+u^2} = \lim_{b\to\infty} \int_0^b \frac{du}{1+u^2}$

$= \lim_{b\to\infty} \left(\tan^{-1} u \Big|_0^b\right) = \lim_{b\to\infty} \left(\tan^{-1} b - \tan^{-1} 0\right) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$ Converges.

So, by the integral test $\sum_{n=1}^{\infty} \frac{1}{n(1+\ln^2 n)}$ converges.

$\boxed{38}$ $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$.

let $f(x) = \frac{x}{x^2+1}$

$f'(x) = \frac{(x^2+1) - 2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$

f' sign

So, $f(x)$ is Continuous, positive & decreasing $\forall x \geq 1$.

$\int_1^{\infty} \frac{x}{x^2+1} = \lim_{b\to\infty} \int_1^b \frac{x}{x^2+1}$

$= \lim_{b\to\infty} \left(\frac{1}{2} \ln |x^2+1| \Big|_1^b \right) = \lim_{b\to\infty} \left(\frac{1}{2} \ln (b^2+1) - \frac{1}{2} \ln 2 \right)$

$= \infty$ diverges.

So, by integral test the series diverges

$\boxed{40}$ $\sum_{n=1}^{\infty} \operatorname{sech}^2 n$.

let $f(x) = \operatorname{sech}^2 x = \frac{1}{\cosh^2 x} = \left(\frac{2}{e^x + e^{-x}} \right)^2 = \frac{4}{(e^x + e^{-x})^2}$

$f'(x) = - \frac{8(e^x - e^{-x})}{(e^x + e^{-x})^3}$

So, $f(x)$ is positive, continuous & decreasing $\forall x \geq 1$.

Now, $\int_1^{\infty} \operatorname{sech}^2 x \, dx = \lim_{b\to\infty} \int_1^b \operatorname{sech}^2 x \, dx$.

$= \lim_{b\to\infty} \left(\tanh x \Big|_1^b \right) = \lim_{b\to\infty} (\tanh b - \tanh 1)$

$= 1 - \tanh 1$

converges.

So, by the integral test

$\sum_{n=1}^{\infty} \operatorname{sech}^2 n$ converges.

remember:-

$\tanh b = \frac{e^b - e^{-b}}{e^b + e^{-b}}$

$\lim_{b\to\infty} \frac{1 - \frac{e^{-b}}{e^b}}{1 + \frac{e^{-b}}{e^b}} = \frac{1-0}{1+0} = \boxed{1}$

42) For what values of a do the series converge??

$$\sum_{n=3}^{\infty}\left(\frac{1}{n-1}-\frac{2a}{n+1}\right)$$

by using the integral test :-

$$\int_3^{\infty}\left(\frac{1}{x-1}-\frac{2a}{x+1}\right)dx=\lim_{b\to\infty}\int_3^{b}\left(\frac{1}{x-1}-\frac{2a}{x+1}\right)dx.$$

$$=\lim_{b\to\infty}\left(\ln|x-1|-2a\ln|x+1|\Big|_3^b\right)$$

$$=\lim_{b\to\infty}\left(\ln\left|\frac{x-1}{(x+1)^{2a}}\right|\Big|_3^b\right)$$

$$=\lim_{b\to\infty}\left(\ln\left|\frac{b-1}{(b+1)^{2a}}\right|-\ln\left|\frac{2}{4^{2a}}\right|\right)$$

$$\lim_{b\to\infty}\ln\left(\frac{b-1}{(b+1)^{2a}}\right)=\lim_{b\to\infty}\ln\frac{1}{2a(b+1)^{2a-1}}$$

$$=\begin{cases}0 & , a=\frac{1}{2}\\ \infty & , a<\frac{1}{2}\end{cases}$$

If $a>\frac{1}{2}$, the terms of the series become negative & the integral test does not apply.

* but when $a>\frac{1}{2}$, the series behaves like a negative multiple of the harmonic series, & so, it's diverges.

So, $\sum_{n=3}^{\infty}\frac{1}{n-1}-\frac{2a}{n+1}$ converges only when $\underline{a=\frac{1}{2}}$

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10.4 :- Comparsion Tests

$\boxed{8}$ Use D.C.T to determine if the following sequence converges or diverges.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{\sqrt{n^2+3}}$$

Compare with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

note that :- $\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{\sqrt{n^2+3}}$ & $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ have nonnegative terms.

$\Rightarrow$ for $n \geq 1 \Rightarrow \sqrt{n} \geq 1$

$$2\sqrt{n} \geq 2$$

$$(2\sqrt{n}+1 \geq 3) \times n.$$

$$2n\sqrt{n}+n \geq 3n \geq 3$$ add n^2 for both sides.

$$2n\sqrt{n}+n+n^2 \geq n^2+3$$

$$n(n+2\sqrt{n}+1) \geq n^2+3.$$

$$n(\sqrt{n}+1)^2 \geq n^2+3$$

$$\frac{n(\sqrt{n}+1)^2}{n^2+3} \geq 1$$ take the square root for the both sides.

$$\frac{\sqrt{n}(\sqrt{n}+1)}{\sqrt{n^2+3}} \geq 1.$$

$$\frac{\sqrt{n}+1}{\sqrt{n^2+3}} \geq \frac{1}{\sqrt{n}}$$ (both series have nonnegative terms for $n \geq 1$).

but, we know that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series, since $p = \frac{1}{2} \leq 1$

So, by D.C.T $\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{\sqrt{n^2+3}}$ diverges.

(15) Use L.C.T to determine if the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{\ln n}$$

☆ Compare with $\sum_{n=2}^{\infty} \frac{1}{n}$, which is divergent p-series, since $p = 1 \leq 1$.

☆ $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ & $\sum_{n=2}^{\infty} \frac{1}{n}$ have positive terms for $n \geq 2$.

☆ Now, $\lim_{n\to\infty} \frac{\frac{1}{\ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\ln n} = \lim_{n\to\infty} \frac{1}{1/n} = \infty$.

$\Rightarrow$ by L.C.T $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges.

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(18) Use any method to determine if the following series converges or diverges

$$\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$$

☆ Let use L.C.T & compare with $\sum_{n=1}^{\infty} \frac{1}{n}$

☆ Both series have positive terms.

☆ $\sum_{n=1}^{\infty} \frac{1}{n}$ divergent (harmonic series).

☆ $\lim_{n\to\infty} \frac{\frac{3}{n+\sqrt{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{3n}{n+\sqrt{n}} = 3 > 0$.

So, both series converge or both diverge.

but, we say that, $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent

So, by L.C.T $\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$ is also divergent.

By D.C.T

$n+n+n > n+\sqrt{n}+0$.

$3n > n+\sqrt{n}$.

$\frac{3}{n+\sqrt{n}} > \frac{1}{n}$

$\sum \frac{1}{n}$ div

So by D.C.T $\sum \frac{3}{n+\sqrt{n}}$ div.

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27) $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$

Compare with $\sum_{n=3}^{\infty} \frac{1}{n}$ Note that: $\sum_{n=3}^{\infty} \frac{1}{n}$ & $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$ have positive terms.

$\Rightarrow n > \ln n$ take ln for both sides.

$\ln n > \ln(\ln n)$

& $n > \ln n > \ln(\ln n)$.

So, $\frac{1}{n} < \frac{1}{\ln(\ln n)}$

$\sum_{n=3}^{\infty} \frac{1}{n} < \sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$

but, we know that $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges (harmonic series).

So, by D.C.T $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$ diverges.

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28) $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$.

Let use L.C.T & compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Note: $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$ & $\sum_{n=1}^{\infty} \frac{1}{n^2}$ have nonnegative terms

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (P-series, $P > 1$).

Now, $\lim_{n \to \infty} \frac{\frac{(\ln n)^2}{n^3}}{\frac{1}{n^2}} = \lim_{n \to \infty} \frac{(\ln n)^2}{n^3} \cdot n^2 = \lim_{n \to \infty} \frac{(\ln n)^2}{n}$ $\frac{\infty}{\infty}$

$$= \lim_{n\to\infty} \frac{2 \ln n \cdot \left(\frac{1}{n}\right)}{1} = \lim_{n\to\infty} \frac{2 \ln n}{n} \quad \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{2\left(\frac{1}{n}\right)}{1} = \lim_{n\to\infty} \frac{2}{n} = 0.$$

So, by L.C.T $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$ converges.

$\boxed{32}$ $\sum_{n=2}^{\infty} \frac{\ln(n+1)}{n+1}$

Use L.C.T & compare with $\sum_{n=2}^{\infty} \frac{1}{n+1}$

$$\lim_{n\to\infty} \frac{\frac{\ln(n+1)}{n+1}}{\frac{1}{n+1}} = \lim_{n\to\infty} \ln(n+1) = \infty.$$

$\sum_{n=2}^{\infty} \frac{1}{n+1}$ (use the integral test). $\Rightarrow f(x) = \frac{1}{x+1}$ (+ve, cont. & decreasing)

↳ have positive terms.

Now, $\int_2^{\infty} \frac{1}{x+1}\,dx = \lim_{b\to\infty} \int_2^b \frac{dx}{x+1} = \lim_{b\to\infty} \left(\ln|x+1| \Big|_2^b\right)$

$$= \lim_{b\to\infty} \left(\ln|b+1| - \ln 3\right)$$

$$= \infty \quad \text{diverges.}$$

So, $\sum_{n=2}^{\infty} \frac{1}{n+1}$ diverges by integral test.

So, by L.C.T $\sum_{n=2}^{\infty} \frac{\ln(n+1)}{n+1}$ diverges

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Note: You can use the integral test to show that $\sum_{n=2}^{\infty} \frac{\ln(n+1)}{n+1}$ diverges.

explain: $\int_2^{\infty} \frac{\ln(x+1)}{x+1}\,dx = \int_{\ln 3}^{\infty} u\,du$

$$= \lim_{b\to\infty} \left[\frac{1}{2}u^2\right]_{\ln 3}^{b} = \lim_{b\to\infty} \frac{1}{2}\left(b^2 - \ln^2 3\right) = \infty$$

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$$\sum_{n=1}^{\infty} \frac{2^n + 3^n}{3^n + 4^n}$$

Let Compare with $\sum_{n=1}^{\infty} \frac{2^n+3^n}{4^n}$ & use D.C.T.

* $\sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+4^n}$ & $\sum_{n=1}^{\infty} \frac{2^n+3^n}{4^n}$ have nonnegative terms.

Now, $\frac{2^n+3^n}{3^n+4^n} < \frac{2^n+3^n}{4^n}$

$$\sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+4^n} < \sum_{n=1}^{\infty} \frac{2^n+3^n}{4^n}.$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{2^n+3^n}{4^n} = \sum_{n=1}^{\infty} \left[\left(\frac{2}{4}\right)^n + \left(\frac{3}{4}\right)^n\right]$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$$

(Geometric serieses with $|r|<1$)

$$= \frac{\frac{1}{2}}{1-\frac{1}{2}} + \frac{\frac{3}{4}}{1-\frac{3}{4}}$$

$$= \frac{\frac{1}{2}}{\frac{1}{2}} + \frac{\frac{3}{4}}{\frac{1}{4}} = 1+3 = \boxed{4}$$

So, $\sum_{n=1}^{\infty} \frac{2^n+3^n}{4^n}$ Converges.

By D.C.T $\sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+4^n}$ converges.

Note :- We can use L.C.T

$$\lim_{n\to\infty} \frac{\frac{2^n+3^n}{3^n+4^n}}{\frac{2^n+3^n}{4^n}} = \lim_{n\to\infty} \frac{4^n}{3^n+4^n} = 1 > 0$$ (both converges)

43 $\sum_{n=2}^{\infty} \frac{1}{n!}$

$n! = n(n-1)(n-2)\cdots 2\cdot 1$

$n! \geq n(n-1)$

$\frac{1}{n!} \leq \frac{1}{n(n-1)} = \frac{1}{n^2-n}$

$\sum_{n=2}^{\infty} \frac{1}{n!} \leq \sum_{n=2}^{\infty} \frac{1}{n^2-n}$

Compare with

$\sum_{n=2}^{\infty} \frac{1}{n^2-n}$ & use D.C.T.

$\sum_{n=2}^{\infty} \frac{1}{n!} \leq \sum_{n=2}^{\infty} \frac{1}{n^2-n}$

$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n^2-n}$ converges.

So, $\sum_{n=2}^{\infty} \frac{1}{n!}$ converges.

Now: Determine $\sum_{n=2}^{\infty} \frac{1}{n^2-n}$ converges or diverges.

→ Compare with $\sum_{n=2}^{\infty} \frac{1}{n^2}$.

($\sum_{n=2}^{\infty} \frac{1}{n^2-n}$ & $\sum_{n=2}^{\infty} \frac{1}{n^2}$ have nonnegative terms.)

& $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges (p-series, $p>1$).

by L.C.T

$$\lim_{n\to\infty} \frac{\frac{1}{n^2-n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-n} = 1 > 0$$

So, both converges.

$\sum_{n=2}^{\infty} \frac{1}{n^2-n}$ converges.

Note: You can use the ratio test

52) $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$

Compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, $\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right.$ converges, p-series with $p>1$).

$\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$ & $\sum_{n=1}^{\infty} \frac{1}{n^2}$ have nonnegative terms.

$$\lim_{n\to\infty} \frac{\frac{\sqrt[n]{n}}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{n^2} \cdot n^2$$

$$= \lim_{n\to\infty} \sqrt[n]{n} = 1 > 0$$

So, by L.C.T $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$ converges.