10.5 The ratio & Root tests.

$\boxed{7}$ Use the ratio test to determine if the following series converges or diverges

$$\sum_{n=1}^{\infty} \frac{n^2 (n+2)!}{n!\, 3^{2n}}$$

$\frac{n^2 (n+2)!}{n!\, 3^{2n}} > 0$ for all $n \geq 1$.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left[\frac{\frac{(n+1)^2 ((n+1)+2)!}{(n+1)!\, 3^{2(n+1)}}}{\frac{n^2 (n+2)!}{n!\, 3^{2n}}} \right]$$

$$= \lim_{n\to\infty} \frac{(n+1)^2 (n+3)!}{(n+1)!\, 3^{2n+2}} \cdot \frac{n!\, 3^{2n}}{n^2 (n+2)!}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2 (n+3).\cancel{(n+2)!} \cdot \cancel{n!}\, \cancel{3^{2n}}}{(n+1).\cancel{n!}\, \cancel{3^{2n}} . 3^2 . n^2 \cancel{(n+2)!}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2 (n+3)}{9(n+1)\, n^2}$$

$$= \boxed{\frac{1}{9}} < 1$$

by ration test, the series is convergent.

(12) $\sum_{n=1}^{\infty} \left(\ln\left(e^2 + \frac{1}{n}\right) \right)^{n+1}$ use root test.

$\left[\ln\left(e^2+\frac{1}{n}\right)\right]^{n+1} \geq 0$ for all $n \geq 1$

Now, $\lim_{n\to\infty} \sqrt[n]{\left[\ln\left(e^2+\frac{1}{n}\right)\right]^{n+1}}$

$= \lim_{n\to\infty} \left[\ln\left(e^2+\frac{1}{n}\right)\right]^{\frac{n+1}{n}}$

$= \lim_{n\to\infty} \left[\ln\left(e^2+\frac{1}{n}\right)\right]^{1+\frac{1}{n}}$

$= \left[\ln\left(e^2+0\right)\right]^{1+0}$

$= 2 > 1$

by The root test the series diverges.

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(16) $\sum_{n=2}^{\infty} \frac{1}{n^{1+n}}$, use root test.

$\frac{1}{n^{1+n}} \geq 0$ for all $n \geq 2$.

$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n^{1+n}}} = \lim_{n\to\infty} \frac{\sqrt[n]{1}}{\sqrt[n]{n^{1+n}}}$

$= \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}} \cdot n^{\frac{n}{n}}}$

$= \lim_{n\to\infty} \frac{1}{\sqrt[n]{n} \cdot n} = 0 < 1$

So, by The root test, the series converges.

20 Use any method to determine if the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n!}{10^n}$$

let use the ratio test.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{10^{n+1}}}{\frac{n!}{10^n}}$$

$$= \lim_{n\to\infty} \frac{(n+1).\cancel{n!}}{\cancel{10^n}.10} \cdot \frac{\cancel{10^n}}{\cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{(n+1)}{10} = \infty > 1$$

The series diverges by the ratio test.

28

$$\sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$$

by root test, $\lim_{n\to\infty} \sqrt[n]{\frac{(\ln n)^n}{n^n}} = \lim_{n\to\infty} \frac{\ln n}{n}$ $\frac{\infty}{\infty}$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0 < 1$$

so, The series converges.

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38) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Let use ratio test :>

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left[\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} \right]$$

$$= \lim_{n\to\infty} \frac{\cancel{(n+1)}\,\cancel{n!}}{(n+1)^n \cancel{(n+1)}} \cdot \frac{n^n}{\cancel{n!}} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n}$$

$$= \lim_{n\to\infty} \frac{1}{\frac{(n+1)^n}{n^n}} = \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e^1}$$

$$= \frac{1}{e} < 1$$

The series converges.

---

43) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

using ratio test :>

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{((n+1)!)^2}{(2(n+1))!} \cdot \frac{(2n)!}{(n!)^2}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2 \cancel{(n!)^2}}{(2n+2)(2n+1)\cancel{(2n)!}} \cdot \frac{\cancel{(2n)!}}{\cancel{(n!)^2}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1$$

So, the series converges.

(46) $a_1 = 1$, $a_{n+1} = \frac{1+\tan^{-1} n}{n} a_n$.

Let use ratio test.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(1+\tan^{-1} n)(a_n)}{n}}{a_n}$$

$$= \lim_{n\to\infty} \frac{(1+\tan^{-1} n)\cancel{a_n}}{n} \cdot \frac{1}{\cancel{a_n}}$$

$$= \frac{1+\frac{\pi}{2}}{\lim_{n\to\infty} n} = 0 < 1$$

∴ The series converges.

---

(60) $\sum_{n=1}^{\infty} \frac{n^n}{(2^n)^2}$

Using root test.

$$\lim_{n\to\infty} \sqrt[n]{\frac{n^n}{(2^n)^2}} = \lim_{n\to\infty} \frac{n^{\frac{n}{n}}}{2^{\frac{2n}{n}}}$$

$$= \lim_{n\to\infty} \frac{n}{2^2}$$

$$= \lim_{n\to\infty} \frac{n}{4}$$

$$= \infty$$

The series diverges.

10.6 Alternating Series, Absolute & Conditional Converges.

8 Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} (-1)^n \cdot \frac{10^n}{(n+1)!}$$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{10^n}{(n+1)!}$$

Using ratio test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{10^{n+1}}{(n+1+1)!} \cdot \frac{(n+1)!}{10^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{10^n} \cdot 10}{(n+2)\cancel{(n+1)!}} \cdot \frac{\cancel{(n+1)!}}{\cancel{10^n}}$$

$$= \lim_{n\to\infty} \frac{10}{n+2} = 0 < 1$$

So, $\sum_{n=1}^{\infty} |a_n|$ converges

The series converges absolutely $\Rightarrow \sum_{n=1}^{\infty} (-1)^n \frac{10^n}{(n+1)!}$ converges

by the Absolute convergence test.

20 $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n!}{2^n}$

by nth term test

$$\lim_{n\to\infty} \frac{n!}{2^n} = \frac{1}{\lim_{n\to\infty}\left(\frac{2^n}{n!}\right)} = \frac{1}{0} = \infty$$

The series diverges.

13 $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}+1}{n+1}$

Using the alternating series test :-

1] $U_n = \frac{\sqrt{n}+1}{n+1} > 0 \quad , \forall n \geq 1.$

2] $U_n \geq U_{n+1}$

let $f(x) = \frac{\sqrt{x}+1}{x+1}$, $x \geq 1.$

$$f'(x) = \frac{\frac{x+1}{2\sqrt{x}} - (\sqrt{x}+1)}{(x+1)^2}$$

$$= \frac{x+1-2x-2\sqrt{x}}{2\sqrt{x}\,(x+1)^2}$$

$$= \frac{-x-2\sqrt{x}+1}{2\sqrt{x}\,(x+1)^2} < 0 \;, \forall x \geq 1$$

$f(x)$ is decreasing

3] $\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{\sqrt{n}+1}{n+1} = \boxed{0}$

So, the series converges by alternating series test.

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(25) Determine if the series converges absolutely, conditionally or diverges.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1+n}{n^2}$$

★ Converges Absolutely ??

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1+n}{n^2}$$

$$= \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\substack{\text{converges} \\ (\text{P-series } P>1)}} + \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\substack{\text{diverges} \\ (\text{harmonic series})}} = \text{diverges.}$$

⇒ The series does not converge absolutely.

★ Converges Conditionally ??

$\boxed{1}$ let $U_n = \dfrac{1+n}{n^2} > 0$

$f(x) = \dfrac{1+x}{x^2} = \dfrac{1}{x^2} + \dfrac{1}{x}$, $x \geq 1$.

$\boxed{2}$ ⇒ $U_n > U_{n+1} > 0 \quad \forall n \geq 1$.

$f'(x) = -\left(\dfrac{2}{x^3} + \dfrac{1}{x^2}\right) < 0 \quad \forall x \geq 1$.

$\boxed{3}$ & $\lim_{n\to\infty} U_n = \lim_{n\to\infty} \dfrac{1+n}{n^2} = 0$

$f(x)$ is decreasing.

⇒ Converges by alternating series test.

⇒ So, The series converges conditionally but not converges absolutely.

(30) $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n-\ln n}$

★ Converges absolutely?? (No)

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{\ln n}{n-\ln n}$$

let use comparison test & compare with $\frac{1}{n}$.

$n - \ln n < n$.

$$\frac{1}{n-\ln n} > \frac{1}{n} \Rightarrow \frac{\ln n}{n-\ln n} > \frac{1}{n}$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ (diverges harmonic series).

So, by direct comparison test $\sum_{n=1}^{\infty} \frac{\ln n}{n-\ln n}$ diverges.

★ Converges conditionally?? (Yes).

1] $U_n = \frac{\ln n}{n-\ln n} \geq 0 \quad \forall n \geq 1.$

2] $f(x) = \frac{\ln x}{x - \ln x}$, $f'(x) = \dfrac{\frac{x-\ln x}{x} - \ln x\left(1-\frac{1}{x}\right)}{(x-\ln x)^2}$.

$$= \frac{x - \cancel{\ln x} - x\ln x - \cancel{\ln x}}{x\,(x-\ln x)^2}$$

$$= \frac{\cancel{x}(1-\ln x)}{\cancel{x}(x-\ln x)^2} < 0 \quad \forall x > e$$

So, $U_n \geq U_{n+1} \quad \forall n \geq 3$.

3] Now, $\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{\ln n}{n-\ln n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1-\frac{1}{n}} = 0$

$\Rightarrow \sum (-1)^n \frac{\ln n}{n-\ln n}$ (converges by alternating series test).

39 $\sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{2^n \, n! \, n}$

diverges by n-th term test

since $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{(2n)!}{2^n \, n! \, n}$

$$= \lim_{n\to\infty} \frac{\cancel{(2n)} (2n-1)(2n-2)\cdots(n+1)\cancel{n!}}{2^n \, \cancel{n!} \, \cancel{n}}$$

$$= \lim_{n\to\infty} \frac{(2n-1)(2n-2)\cdots(n+1)}{2^{n-1}}$$

$$> \lim_{n\to\infty} \left(\frac{n+1}{2}\right)^{n-1}$$

$$= \infty \neq 0.$$

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42 $\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n^2+n} - n\right)$

Let use n-th term test

$$\lim_{n\to\infty} \left(\sqrt{n^2+n} - n\right) = \lim_{n\to\infty} \left(\sqrt{n^2+n} - n\right) \cdot \frac{\sqrt{n^2+n} + n}{\sqrt{n^2+n} + n}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2} + n - \cancel{n^2}}{\sqrt{n^2+n} + n} = \lim_{n\to\infty} \frac{n}{|n|\sqrt{1+\frac{1}{n}} + n}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}\left(\sqrt{1+\frac{1}{n}} + 1\right)} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}} + 1}$$

$$= \frac{1}{1+1} = \frac{1}{2} \neq 0$$

The series diverges.

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50 Estimate the magnitude of the error involved in using the sum of the first four terms to approximate the sum of the series.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{10^n}$$

$$|error| < u_{4+1} = |a_{4+1}|$$

$$|error| < |a_5| = \left| (-1)^{5+1} \frac{1}{10^5} \right|$$

$$|error| < 0.00001$$

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54 Determine how many terms should be used to estimate the sum of the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$ with an error of less than 0.001

$$|error| < 0.001$$

$$\Rightarrow u_{n+1} < 0.001$$

$$\frac{n+1}{(n+1)^2+1} < 0.001$$

$$(n+1)\,1000 < (n+1)^2 + 1$$

$$1000n + 1000 < n^2 + 2n + 2$$

$$0 < n^2 - 998n - 998$$

$$n > \frac{--998 \mp \sqrt{(-998)^2 - 4(1)(-998)}}{2(1)} \approx 998.99899$$

$$n \geq 999$$