10.7 : Power Series

a Find the series' radius & interval of convergence.

b For what values of x does the series converges absolutely?

c For what values of x does the series converges conditionally?

4 $\displaystyle\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$

Ratio test :⇒ The series converges absolutely if ⇒

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| < 1$$

$$\lim_{n\to\infty} \left| \frac{\frac{(3x-2)^{n+1}}{n+1}}{\frac{(3x-2)^n}{n}} \right| = \lim_{n\to\infty} \left| \frac{\cancel{(3x-2)^n}\,(3x-2)}{n+1} \cdot \frac{n}{\cancel{(3x-2)^n}} \right|$$

$$= \lim_{n\to\infty} \frac{n}{n+1} \, |3x-2|$$

$$= |3x-2| < 1$$

$$-1 < 3x-2 < 1$$

$$1 < 3x < 3$$

$$\boxed{\frac{1}{3} < x < 1}$$

☆ for $x = \frac{1}{3}$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, converges conditionaly by A.S.T

$\left(u_n = \frac{1}{n},\ 1)\ u_n > 0,\ 2)\ u_n \searrow\ 3)\ \lim_{n\to\infty} u_n = 0\right)$.

☆ for $x = 1$: $\displaystyle\sum_{n=1}^{\infty} \frac{1^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$, diverges (harmonic series).

a The radius is $\frac{1}{3}$, the Interval of Convergence is $\frac{1}{3} \leq x < 1$.

b The interval of absolute convergence is $\frac{1}{3} < x < 1$.

c The series converges conditionally at $x = \frac{1}{3}$.

12 $\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$

Ratio test: The series converges absolutely if:

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| < 1.$$

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{3^{n+1} x^{n+1}}{(n+1)!}}{\frac{3^n x^n}{n!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{3^n} \cdot 3 \cancel{x^n} x}{(n+1) \cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{3^n} \cancel{x^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{3x}{n+1} \right|$$

$$= 3|x| \lim_{n\to\infty} \frac{1}{n+1}$$

$$= 0 < 1 \quad \text{for all } x.$$

a the radius is ∞; the series converges for all x.

b the series converges absolutely for all x.

c there are no values for which the series converges conditionally.

$\boxed{14}$ $\sum_{n=1}^{\infty} \frac{(x-1)^n}{3^n n^2}$

Ratio test:

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(x-1)^{n+1}}{3^{n+1}(n+1)^2}}{\frac{(x-1)^n}{3^n n^2}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{(x-1)^n} \cdot (x-1)}{\cancel{3^n} \cdot 3\,(n+1)^2} \cdot \frac{\cancel{3^n}\, n^2}{\cancel{(x-1)^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x-1)\, n^2}{3(n+1)^2} \right|$$

$$= \frac{|x-1|}{3} \lim_{n\to\infty} \frac{n^2}{(n+1)^2}$$

$$= \frac{|x-1|}{3} \cdot 1$$

$$= \frac{|x-1|}{3}.$$

The series converges absolutely when $\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| < 1.$

$\Rightarrow \frac{|x-1|}{3} < 1$

$\Rightarrow |x-1| < 3$

$\Rightarrow -3 < x-1 < 3$

$\boxed{-2 < x < 4}$

☆ When $x=-2$:- we have $\sum_{n=1}^{\infty} \frac{(-3)^n}{3^n n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, an absolutely convergent series.

☆ When $x=4$:- we have $\sum_{n=1}^{\infty} \frac{3^n}{3^n n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$, an absoutely convergent series.

a) The radius is 3; the interval of convergence is $-2 \leq x \leq 4$.

b) The interval of absolute convergence is $-2 \leq x \leq 4$.

c) There are no values for which the series converges conditionally.

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23) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n x^n$

Ratio test: The series converges absolutely when $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$.

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left| \frac{\left(1+\frac{1}{n+1}\right)^{n+1} x^{n+1}}{\left(1+\frac{1}{n}\right)^n x^n} \right|$$

$$= |x| \frac{\lim_{n\to\infty} \left(1+\frac{1}{n+1}\right)^{n+1}}{\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n}$$

$\lim_{n\to\infty} \left(1+\frac{1}{n+1}\right)^{n+1}$

let $L = n+1$.

$n \to \infty$, $L \to \infty$

$\lim_{L\to\infty} \left(1+\frac{1}{L}\right)^L = e^1 = e$

$$= |x| \cdot \frac{e}{e}$$

$$= |x|.$$

$\Rightarrow |x| < 1$.

$-1 < x < 1$.

☆ When $x=-1$ we have $\sum_{n=1}^{\infty} (-1)^n \left(1+\frac{1}{n}\right)^n$:-

a divergent series by nth term test since

$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \neq 0$.

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☆ when $x=1$ we have $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n$:

a divergent series by n-th term test.

a) the radius is 1, the interval of convergence $-1<x<1$.

b) the interval of absolute convergence is $-1<x<1$.

c) there are no values for which the series converges conditionally.

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29 $\sum_{n=2}^{\infty}\frac{x^n}{n(\ln n)^2}$

Ratio test : The series converges absolutely when :

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1.$$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1(\ln(n+1))^2}\cdot\frac{n(\ln n)^2}{x^n}\right|<1.$$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{x\, n(\ln n)^2}{n+1\,(\ln(n+1))^2}\right|<1.$$

$$\Rightarrow |x|\lim_{n\to\infty}\left(\frac{n}{n+1}\right)\left(\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)}\right)^2<1.$$

$$\Rightarrow |x|\cdot(1)\left(\lim_{n\to\infty}\frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n+1}\right)}\right)^2<1.$$

$$\Rightarrow |x|\left(\lim_{n\to\infty}\frac{n+1}{n}\right)^2<1.$$

$$\Rightarrow |x|\,(1)^2<1.$$

$$\Rightarrow |x|<1$$

$$\Rightarrow \boxed{-1<x<1}$$

☆ when $x=-1$, $\sum_{n=2}^{\infty}\frac{(-1)^n}{n(\ln n)^2}$

which converges absolutely by the integral test.

$$\Rightarrow \sum_{n=2}^{\infty}\left|\frac{(-1)^n}{n(\ln n)^2}\right|=\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$$

have positive terms

$\Rightarrow$ let $f(x)=\frac{1}{x(\ln x)^2}$, cont, positive & decreasing $\forall x\geq 2$.

$\Rightarrow \int_2^{\infty}\frac{1}{x(\ln x)^2}dx$, let $u=\ln x$, $du=\frac{dx}{x}$

$$\Rightarrow \lim_{b\to\infty}\int_{\ln 2}^{b}u^{-2}\,du=\lim_{b\to\infty}-\frac{1}{u}\Big]_{\ln 2}^{b}$$

$$=\lim_{b\to\infty}-\frac{1}{b}+\frac{1}{\ln 2}=\frac{1}{\ln 2}\quad\text{(converges)}$$

☆ when $x=1$, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$, converges by the integral test.

[a] the radius is 1, the interval of convergence is $-1 \leq x \leq 1$

[b] the interval of absolute convergence is $-1 \leq x \leq 1$.

[c] there are no values for which the series converges conditionally.

[32] $\sum_{n=1}^{\infty} \frac{(3x+1)^{n+1}}{2n+2}$

by ratio test, the series converges absolutely if :-

$$\Rightarrow \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| < 1.$$

$$\Rightarrow \lim_{n\to\infty} \left| \frac{(3x+1)^{n+2}}{2n+4} \cdot \frac{2n+2}{(3x+1)^{n+1}} \right| < 1.$$

$$\Rightarrow \lim_{n\to\infty} \left| (3x+1) \cdot \frac{2n+2}{2n+4} \right| < 1.$$

$$\Rightarrow |3x+1| \lim_{n\to\infty} \frac{2n+2}{2n+4} < 1.$$

$$\Rightarrow |3x+1| \cdot (1) < 1.$$

$$\Rightarrow |3x+1| < 1.$$

$$\Rightarrow -1 < 3x+1 < 1$$

$$\Rightarrow -2 < 3x < 0$$

$$\Rightarrow \boxed{-\frac{2}{3} < x < 0}$$

☆ when $x = -\frac{2}{3}$, we have $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+2}$,

a conditionally convergent series by A.S.T.

☆ when $x=0$, we have $\sum_{n=1}^{\infty} \frac{1}{2n+2}$

a divergent series.

☆ $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+2}$, $u_n = \frac{1}{2n+2}$

A.S.T :-

1) $u_n > 0$. ✓

2) $u_n > u_{n+1}$. ✓

3) $\lim_{n\to\infty} u_n = 0$ ✓

☆ $\sum_{n=1}^{\infty} \frac{1}{2n+2}$, use L.C.T & compare with $\sum_{n=1}^{\infty} \frac{1}{n}$ (div harmonic series)

$\lim_{n\to\infty} \frac{\frac{1}{2n+2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2n+2} = \frac{1}{2} > 0$ (both div)

(a) the radius is $\frac{1}{3}$, the interval of convergence is $-\frac{2}{3} \leqslant x < 0$.

(b) the interval of absolute convergence is $-\frac{2}{3} < x < 0$.

(c) the series converges conditionally at $x = -\frac{2}{3}$.

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40 Find the series' radius of convergence

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} x^n$$

Root test:- The series converges absolutely if

$$\lim_{n\to\infty} \sqrt[n]{|u_n|} < 1$$

$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n}{n+1}\right)^{n^2} x^n\right|} < 1.$$

$$\Rightarrow \lim_{n\to\infty} \left|\left(\frac{n}{n+1}\right)^{\frac{n^2}{n}} x^{\frac{n}{n}}\right| < 1.$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n |x| < 1.$$

$$\Rightarrow |x| \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n} < 1.$$

$$\Rightarrow |x| \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} < 1.$$

$$\Rightarrow |x| \cdot \frac{1}{e^1} < 1.$$

$$\Rightarrow |x| < e$$

$$\Rightarrow -e < x < e.$$

So, The radius of convergence $R = e$.

46. Use theorem (20) – in the textbook – to find the series' interval of convergence & within this interval, the sum of the series as a function

$$\sum_{n=0}^{\infty} (\ln x)^n$$

Theorem (20): If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$, then

$\sum_{n=0}^{\infty} a_n (f(x))^n$ converges absolutely for any continuous function

f on $|f(x)| < R$.

by Ratio test: $\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| < 1$.

⇒ The series converges absolutely if

$$\Rightarrow \lim_{n\to\infty} \left| \frac{(\ln x)^{n+1}}{(\ln x)^n} \right| < 1.$$

$$\Rightarrow \lim_{n\to\infty} |\ln x| < 1$$

$$\Rightarrow |\ln x| < 1.$$

$$\Rightarrow -1 < \ln x < 1.$$

$$\Rightarrow e^{-1} < x < e$$

* when $x = e^{-1}$ we have $\sum_{n=0}^{\infty} (\ln(e^{-1}))^n = \sum_{n=1}^{\infty} (-1)^n$ which is a divergent series.

* when $x = e$ we have $\sum_{n=0}^{\infty} (\ln e)^n = \sum_{n=0}^{\infty} 1^n$ which is a divergent series.

⇒ the interval of convergence is $e^{-1} < x < e$.

⇒ the sum of the series is $\frac{1}{1-\ln x}$ when $e^{-1} < x < e$

(since $\sum_{n=0}^{\infty} (\ln x)^n$ is a geometric series with $r = \ln x$ & $a = 1$).

49 For what values of x does the series

$1-\frac{1}{2}(x-3)+\frac{1}{4}(x-3)^2+\cdots+\left(-\frac{1}{2}\right)^n(x-3)^n+\cdots$ converge? What is its sum?

What series do you get if you differentiate the given series term by term?

$\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n(x-3)^n$ converges absolutely if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1$ (Ratio test).

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1$$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{\left(\frac{1}{2}\right)^{n+1}(x-3)^{n+1}}{\left(\frac{1}{2}\right)^n(x-3)^n}\right|<1.$$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{1}{2}(x-3)\right|<1.$$

$$\Rightarrow \frac{1}{2}|x-3|<1.$$

$$\Rightarrow |x-3|<2.$$

$$\Rightarrow -2<x-3<2$$

$$\Rightarrow \boxed{1<x<5}$$

When $x=1$, we have $\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n(-2)^n=\sum_{n=0}^{\infty}1^n$ which diverges.

When $x=5$, we have $\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n(2)^n=\sum_{n=0}^{\infty}(-1)^n$ which diverges.

$\Rightarrow$ The interval of convergence is $1<x<5$.

$\Rightarrow$ The sum of this convergent geometric series is $\frac{1}{1+\left(\frac{x-3}{2}\right)}=\frac{1}{\frac{2+x-3}{2}}=\boxed{\frac{2}{x-1}}$

Now, $f(x)=1-\frac{1}{2}(x-3)+\frac{1}{4}(x-3)^2+\cdots+\left(-\frac{1}{2}\right)^n(x-3)^n+\cdots=\frac{2}{x-1}$

$f'(x)=0-\frac{1}{2}+\frac{1}{2}(x-3)+\cdots+\left(-\frac{1}{2}\right)^n n(x-3)^{n-1}+\cdots=-\frac{2}{(x-1)^2}$

is convergent when $1<x<5$, & diverges when $x=1$ or 5.

Note that: the sum for $f'(x)$ is $\frac{-2}{(x-1)^2}$, the derivative of $\frac{2}{x-1}$.