10.8 Taylor & Maclaurin Series

3) Find the taylor Polynomials of orders 0, 1, 2 & 3 generated by f at a.

$f(x) = \ln x \quad , \; a = 1.$

$f(1) = \ln 1 = 0.$

$f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1.$

$f''(x) = -\frac{1}{x^2} \Rightarrow f''(1) = -1$

$f'''(x) = +\frac{2}{x^3} \Rightarrow f'''(1) = 2$

$P_0(x) = f(1) = 0$

$$\begin{aligned} P_1(x) &= f(1) + f'(1)(x-1) \\ &= 0 + 1(x-1) \\ &= x - 1 \end{aligned}$$

$$\begin{aligned} P_2(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 \\ &= 0 + 1(x-1) + \frac{-1}{2}(x-1)^2 \\ &= (x-1) - \frac{1}{2}(x-1)^2 \end{aligned}$$

$$\begin{aligned} P_3(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 \\ &= 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{6}(x-1)^3 \\ &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} \end{aligned}$$

14) Find the Maclaurin series for the function

$$f(x) = \frac{2+x}{1-x}$$

Maclaurin series generated by f is

$$f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$f(0) = 2$

$f'(x) = \frac{3}{(1-x)^2} \Rightarrow f'(0) = 3$

$f''(x) = \frac{6}{(1-x)^3} \Rightarrow f''(0) = 6$

$f'''(x) = \frac{18}{(1-x)^4} \Rightarrow f'''(0) = 18.$

$\vdots$

The Taylor series is

$$2 + 3x + \frac{6}{2!}(x^2) + \frac{18}{3!}x^3 + \cdots$$

$$= 2 + 3x + 3x^2 + 3x^3 + \cdots$$

$$= 2 + \sum_{n=1}^{\infty} 3x^n.$$

20 Find the Maclaurin series for the function

$$f(x) = \sinh x = \frac{e^x - e^{-x}}{2}.$$

$$f(0) = \sinh(0) = \frac{e^0 - e^{-0}}{2} = 0.$$

$$f'(x) = \frac{e^x + e^{-x}}{2} \Rightarrow f'(0) = \frac{1+1}{2} = 1$$

$$f''(x) = \frac{e^x - e^{-x}}{2} \Rightarrow f''(0) = 0$$

$$f'''(x) = \frac{e^x + e^{-x}}{2} \Rightarrow f'''(0) = 1$$

$\vdots$

Maclaurin series $\Rightarrow f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$

$$\sinh x = 0 + 1(x) + \frac{0\,(x^2)}{2} + \frac{1}{3!}x^3 + \cdots$$

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

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22) Find the Maclaurin series of the function

$$f(x) = \frac{x^2}{x+1}$$

$f(0) = 0$

$f'(x) = \frac{x^2+2x}{(x+1)^2} \Rightarrow f'(0) = 0$

$f''(x) = \frac{2}{(x+1)^3} \Rightarrow f''(0) = 2.$

$f'''(x) = \frac{-6}{(x+1)^4} \Rightarrow f'''(0) = -6$

$f^{(n)}(x) = \frac{(-1)^n \, n!}{(x+1)^{n+1}} \Rightarrow f^{(n)}(0) = (-1)^n \, n!$, If $n \geq 2$

Maclaurin series of $f = 0 + 0 + \frac{2}{2!}x^2 + \frac{-6}{3!}x^3 + \cdots$

$$= x^2 - x^3 + x^4 - x^5 + \cdots$$

$$= \sum_{n=2}^{\infty} (-1)^n x^n.$$

27) Find the Taylor series generated by f at $x=a$.

$f(x) = \frac{1}{x^2}$, $a=1$.

$f(1) = 1$

$f'(x) = -\frac{2}{x^3} \Rightarrow f'(1) = -2$

$f''(x) = \frac{6}{x^4} = \frac{3!}{x^4} \Rightarrow f''(1) = 3!$

$f'''(x) = \frac{-4!}{x^5} \Rightarrow f'''(1) = -4!$

$f^n(x) = \frac{(-1)^n(n+1)!}{x^{n+2}} \Rightarrow f^n(1) = (-1)^n (n+1)!$

Taylor series is $f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \cdots$

$= 1 - 2(x-1) + \frac{3!}{2!}(x-1)^2 - \frac{4!}{3!}(x-1)^3 + \cdots$

$= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + \cdots$

$= \sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n$

32. Find the Taylor series generated by

$$f(x) = \sqrt{x+1} \quad , \quad a=0$$

$$f(0) = 1$$

$$f'(x) = \frac{1}{2\sqrt{x+1}} \Rightarrow f'(0) = \frac{1}{2}.$$

$$f''(x) = \frac{-1}{4(x+1)^{3/2}} \Rightarrow f''(0) = -\frac{1}{4}.$$

$$f'''(x) = \frac{3}{8(x+1)^{5/2}} \Rightarrow f'''(0) = \frac{3}{8}$$

$$f^{(4)}(x) = \frac{-15}{16(x+1)^{7/2}} \Rightarrow f^{(4)}(1) = \frac{-15}{16}$$

$$\vdots$$

Taylor series : $f(0) + f'(0)(x-0) + \frac{f''(0)(x-0)^2}{2!} + \frac{f'''(0)(x-0)^3}{3!} + \cdots$

$$\sqrt{x+1} = 1 + \frac{1}{2}x + -\frac{1}{4(2!)}x^2 + \frac{3}{8(3!)}x^3 + \frac{-15}{16(4!)}x^4 + \cdots$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5}{128}x^4 + \cdots$$

37 Use the Taylor series generated by e^x at $x=a$ to show that:

$$e^x = e^a\left[1+(x-a)+\frac{(x-a)^2}{2!}+\cdots\right].$$

$f(a) = e^a$

$f'(x) = e^x \Rightarrow f'(a) = e^a.$

$f''(x) = e^x \Rightarrow f''(a) = e^a$

$f'''(x) = e^x \Rightarrow f'''(a) = e^a.$

$\vdots$

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

* $f(x) = e^x$ & * $f^n(a) = e^a$ for all $n = 0, 1, 2, 3, \ldots$

$$e^x = \frac{e^a (x-a)^0}{0!} + \frac{e^a (x-a)}{1!} + \frac{e^a (x-a)^2}{2!} + \frac{e^a (x-a)^3}{3!} + \cdots$$

$$= e^a\left[1+(x-a)+\frac{(x-a)^2}{2!}+\frac{(x-a)^3}{3!}+\cdots\right]. \text{ at } x=a$$