10.9 Convergence of Taylor series.

10 Use substitution to find the Taylor series at $x=0$ of the function $\frac{1}{2-x}$.

$$\frac{1}{2-x} = \frac{1}{2\left(1-\frac{x}{2}\right)}$$

$$= \frac{1}{2}\left[\frac{1}{1-\frac{x}{2}}\right].$$

Taylor series at $a=0$ of $\frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$

Notice that:- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ { geometric series with $a=1$ & $r=x$

So, by substitution we have:-

$$\frac{1}{1-\frac{x}{2}} = \sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$$

Now, $\frac{1}{2-x} = \frac{1}{2}\left[\frac{1}{1-\frac{x}{2}}\right]$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}}$$

$$= \frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \cdots.$$

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$\boxed{12}$ Use power series operations to find the Taylor series at $x=0$ for the function $\boxed{x^2 \sin x}$

☆ Remember:

Taylor series generated by $\boxed{\sin x}$ at $x=0$ is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots .$$

$$\boxed{x^2 \sin x} = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+1)!}$$

$$= x^3 - \frac{x^5}{3!} + \frac{x^7}{5!} - \frac{x^9}{7!} + \cdots .$$

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18) Use power series operations to find the Taylor series at $x=0$ for the function $\sin^2 x$.

we know that :-

★ Remember :-

$\cos(2x) = 1 - 2\sin^2 x$.

$$\boxed{\sin^2 x = \frac{1}{2} - \frac{\cos(2x)}{2}}$$

$$\Rightarrow \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\Rightarrow \cos(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} x^{2n}}{(2n)!}$$

$$\Rightarrow -\frac{1}{2}\cos(2x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{2^{2n-1} x^{2n}}{(2n)!} = -\frac{1}{2} + \frac{2x^2}{2!} - \frac{2^3 x^4}{4!} + \cdots$$

$$\Rightarrow \boxed{\sin^2 x} = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

$$= \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} + \frac{2x^2}{2!} - \frac{2^3 x^4}{4!} + \frac{2^5 x^6}{6!} - \cdots$$

$$= \boxed{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^{2n-1} x^{2n}}{(2n)!}}$$

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(28) Use power series operations to find the Taylor series at $x=0$ for the function $\ln(x+1) - \ln(1-x)$.

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n$$

Remember $\Rightarrow$ $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.

$$\Rightarrow \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$= 1 - x + x^2 - x^3 + \cdots$$

$$\int \frac{1}{1+x}\,dx = \int (1 - x + x^2 - x^3 + \cdots)\,dx.$$

$$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots}. \quad \star$$

$$\Rightarrow \& \quad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

$$\int \frac{1}{1-x}\,dx = \int (1 + x + x^2 + x^3 + \cdots)\,dx.$$

$$\boxed{-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots} \quad \star\star$$

Now, $\star + \star\star$

$$\boxed{\ln(1+x) - \ln(1-x)} = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \cdots$$

$$\boxed{= \sum_{n=0}^{\infty} 2\frac{x^{2n+1}}{2n+1}}$$

35) Estimate the error if $P_3(x) = x - \frac{x^3}{6}$ is used to estimate the value of $\sin x$ at $x = 0.1$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$P_3(x) = x - \frac{x^3}{3!} = x - \frac{x^3}{6}$$

$f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x.$

$f'''(x) = -\cos x.$

$f^{(4)}(x) = \sin x$

Since $n = 3 \Rightarrow f^{(4)}(x) = \sin x.$

$|f^{(4)}(x)| \leq M$ on $[0, 0.1]$

$|\sin x| \leq 1$ on $[0, 0.1]$

$\Rightarrow$ So, $M = 1$

$$|R_3(0.1)| \leq \frac{M\,|0.1-0|^{3+1}}{(3+1)\,!}$$

$$|R_3(0.1)| \leq \frac{(1)\,|0.1|^4}{4!}$$

$$\leq \frac{0.1^4}{4!} \approx 4.167 \times 10^{-6}$$

Note: You can use alternating series estimation theorem to solve this question.

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37) For approximately what values of x can you replace $\sin x$ by $x - \frac{x^3}{6}$ with an error of magnitude no greater than 5×10^{-4}?

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

By alternating series estimation theorem:

$\Rightarrow$ $|\text{error}| < |\text{first neglected term}|$, then

$$|\text{error}| < \frac{|x^5|}{5!}, \text{ so:}$$

$$\frac{|x^5|}{5!} < 5 \times 10^{-4} \Rightarrow |x|^5 < 5!\,(5 \times 10^{-4}) = 0.06$$

$$\Rightarrow |x| < \sqrt[5]{0.06} \approx 0.5698$$

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$\boxed{41}$ The approximation $e^x = 1 + x + \frac{x^2}{2}$ is used when x is small. Use the Reminder Estimation Theorem to estimate the error when $|x| < 0.1$.

$$e^x = \underbrace{1 + x + \frac{x^2}{2}}_{P_2(x)} \quad \left(\text{Taylor Polynomial of order 2. } n=2\right)$$

$$f^{(n+1)}(x) = f^{3}(x) = e^x ,$$

but $|x| < 0.1 \Rightarrow |f^{(3)}(x)| < |e^x| = e^x < e^{0.1} < 3^{0.1}$

Hence, By remainder estimation theorem.

$$|R_2(x)| < \frac{3^{0.1}|x|^3}{3!}$$

$$< \frac{3^{0.1}(0.1)^3}{3!} = 1.87 \times 10^{-4}$$

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10.10 : The Binomial series & Applications of Taylor Series.

10 Find the first four terms of the binomial series for the function $\dfrac{x}{\sqrt[3]{1+x}}$

remember:

The Binomial series for $(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k$

$$\binom{m}{k} = \frac{m(m-1)(m-2)\dots(m-k+1)}{k!}$$

$$\frac{x}{\sqrt[3]{1+x}} = x(1+x)^{-\frac{1}{3}} \quad , m = -\frac{1}{3}$$

$$= x\left[1 + \sum_{k=1}^{\infty} \binom{-\frac{1}{3}}{k} x^k\right]$$

$$= x\left[1 + \binom{-\frac{1}{3}}{1} x^1 + \binom{-\frac{1}{3}}{2} x^2 + \binom{-\frac{1}{3}}{3} x^3 + \binom{-\frac{1}{3}}{4} x^4 + \dots\right]$$

$$= x\left[1 + -\frac{1}{3}x + \frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{2} x^2 + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{1}{3}-3+1\right)}{3!} x^3 + \dots\right]$$

$$= x - \frac{x^2}{3} + \frac{4}{9(2)} x^3 + \frac{\left(\frac{4}{9}\right)\left(-\frac{7}{3}\right)}{6} x^4 + \dots]$$

$$\frac{x}{\sqrt[3]{1+x}} = x - \frac{x^2}{3} + \frac{2}{9}x^3 - \frac{14}{81}x^4 + \dots .$$

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16) Use the series to estimate the integral's values with an error of magnitude less than 10^{-3}.

$$\int_0^{0.2} \frac{e^{-x}-1}{x}\,dx.$$

we know, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\Rightarrow e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

$$\Rightarrow \frac{e^{-x}-1}{x} = \frac{1}{x}\left(e^{-x}-1\right) = \frac{1}{x}\left(\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!} - 1\right).$$

$$= \frac{1}{x}\left(\left(\cancel{1} - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right) - \cancel{1}\right)$$

$$= \frac{1}{x}\left(-x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right)$$

$$= -1 + \frac{x}{2!} - \frac{x^2}{3!} + \frac{x^3}{4!} - \cdots$$

$$\Rightarrow \int_0^{0.2} \frac{e^{-x}-1}{x}\,dx = \int_0^{0.2}\left(-1 + \frac{x}{2!} - \frac{x^2}{3!} + \frac{x^3}{4!} - \cdots\right)dx.$$

$$= -x + \frac{x^2}{2.2!} - \frac{x^3}{3.3!} + \frac{x^4}{4.4!} - \cdots \Big|_0^{0.2}$$

$$= -0.2 + \frac{(0.2)^2}{2.2!} - \frac{(0.2)^3}{3.3!} + \frac{(0.2)^4}{4.4!} - \cdots$$

Now, with an error of magnitude less than 10^{-3} ($|error| < 10^{-3}$), we find the first term to be numerically less than 10^{-3}:

$$\frac{(0.2)^3}{3.3!} = \frac{0.008}{18} = 4.\bar{4}\times 10^{-4} < 10^{-3}$$

So, the first neglected term is $\frac{(0.2)^3}{3 \cdot 3!}$.

$$\int_0^{0.2} \left(\frac{e^{-x} - 1}{x} \right) dx \approx -(0.2) + \frac{(0.2)^2}{4}$$ [The estimated value of the integral]

26) Find a polynomial that will approximate $F(x)$ throughout the given interval with an error of magnitude less than 10^{-3}.

$$F(x) = \int_0^x t^2 e^{-t^2} dt, \quad [0, 1].$$

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \Rightarrow e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}$$

$$t^2 e^{-t^2} = t^2 \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+2}}{n!}$$

$$t^2 e^{-t^2} = t^2 - t^4 + \frac{t^6}{2!} - \frac{t^8}{3!} + \frac{t^{10}}{4!} - \frac{t^{12}}{5!} + \cdots$$

$$\Rightarrow F(x) = \int_0^x t^2 e^{-t^2} dt = \int_0^x \left(t^2 - t^4 + \frac{t^6}{2!} - \frac{t^8}{3!} + \frac{t^{10}}{4!} - \frac{t^{12}}{5!} + \cdots \right) dt$$

$$= \frac{t^3}{3} - \frac{t^5}{5} + \frac{t^7}{7 \cdot 2!} - \frac{t^9}{9 \cdot 3!} + \frac{t^{11}}{11 \cdot 4!} - \frac{t^{13}}{13 \cdot 5!} + \cdots \Bigg|_0^x$$

$$= \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7 \cdot 2!} - \frac{x^9}{9 \cdot 3!} + \frac{x^{11}}{11 \cdot 4!} - \frac{x^{13}}{13 \cdot 5!} + \cdots.$$

$\Rightarrow$ Now, with an error of magnitude less than 10^{-3}, we find that the first term to be numerically less than 10^{-3} (0.001) is $\frac{1}{13 \cdot 5!}$

$$|\text{error}| < \left| \frac{x^{13}}{13 \cdot 5!} \right| = \frac{|x|^{13}}{13 \cdot 5!} < \frac{1}{13 \cdot 5!} \approx 6.4 \times 10^{-4} < 10^{-3}.$$

$$\Rightarrow \text{So, } F(x) \approx \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7 \cdot 2!} - \frac{x^9}{9 \cdot 3!} + \frac{x^{11}}{11 \cdot 4!}.$$

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Use series to evaluate the Limits :-

30) $$\lim_{x\to 0} \frac{e^x - e^{-x}}{x}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$e^x - e^{-x} = 2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \cdots$$

$$\frac{e^x - e^{-x}}{x} = 2 + \frac{2x^2}{3!} + \frac{2x^4}{5!} + \cdots$$

$$\lim_{x\to 0} \frac{e^x - e^{-x}}{x} = \lim_{x\to 0} \left(2 + \frac{2x^2}{3!} + \frac{2x^4}{5!} + \cdots\right).$$

$$= \boxed{2}$$

33) $$\lim_{y\to 0} \frac{y - \tan^{-1} y}{y^3}$$

$$\tan^{-1} y = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{2n+1}, \quad |y| \leq 1.$$

$$= y - \frac{y^3}{3} + \frac{y^5}{5} - \frac{y^7}{7} + \cdots$$

$$\lim_{y\to 0} \frac{y - \tan^{-1} y}{y^3} = \lim_{y\to 0} \frac{y - y + \frac{y^3}{3} - \frac{y^5}{5} + \frac{y^7}{7} - \cdots}{y^3}$$

$$= \lim_{y\to 0} \left(\frac{1}{3} - \frac{y^2}{5} + \frac{y^4}{7} - \cdots\right)$$

$$\Rightarrow \boxed{\frac{1}{3}}$$

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