11.4 : Graphing in Polar Coordinates.

Identify the symmetric of the curves. Then sketch the curves

1. $r = 1 + \cos\theta$.

a) Check the symmetry about the x-axis.

$(r, -\theta) \Rightarrow r = 1 + \cos(-\theta)$

$= 1 + \cos\theta$

$= r$

$\Rightarrow (r, \theta)$ ✓

Symmetric about the x-axis.

b) Check the symmetry about the y-axis.

$(-r, -\theta) \Rightarrow -r \stackrel{?}{=} 1 + \cos(-\theta)$

$-r \neq 1 + \cos(\theta)$

Cannot tell.

$(r, \pi - \theta) \Rightarrow r \stackrel{?}{=} 1 + \cos(\pi - \theta)$

$\stackrel{?}{=} 1 + \cos(\pi)\cos(\theta) + \sin(\pi)\sin\theta$.

$r \neq 1 - \cos(\theta)$

not symmetric about y-axis.

Therefore not symmetric about the origin. (You can check)

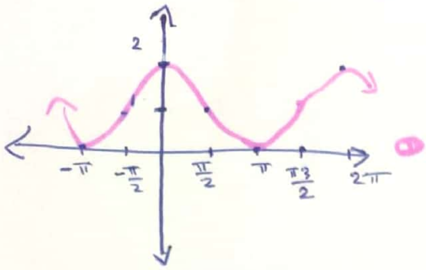
c) Check the symmetry about the origin.

$(-r, \theta) \Rightarrow -r \stackrel{?}{=} 1 + \cos\theta$

$(r, \theta + \pi) \Rightarrow r \stackrel{?}{=} 1 + \cos(\theta + \pi)$

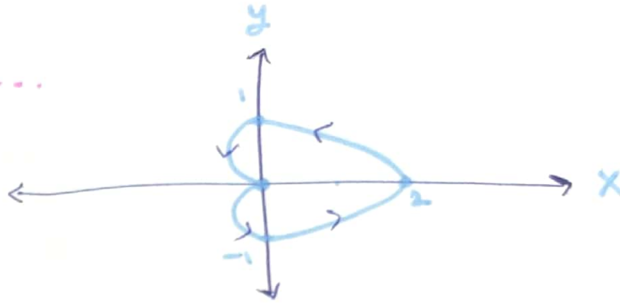
$\stackrel{?}{=} 1 + \cos(\theta)\cos(\pi) - \sin(\theta)\sin(\pi)$

$r \neq 1 - \cos(\theta)$.



$r = 1 + \cos\theta = 0$
$\Rightarrow \theta = \pi, -\pi, 3\pi, \ldots$

$\Longrightarrow$



$(r, \theta) \Rightarrow (x, y)$

s.t. $x = r\cos\theta$
$y = r\sin\theta$.

$(2, 0) \Rightarrow (2, 0)$

$\left(1, \frac{\pi}{2}\right) \Rightarrow (0, 1)$

$(0, \pi) \Rightarrow (0, 0)$

$\left(1, \frac{3\pi}{2}\right) \Rightarrow (0, -1)$

From the graph: No symmetry about y-axis or origin.

$\boxed{6}$ $r = 1 + 2\sin\theta$.

a) x-axis.

$(r, -\theta) \Rightarrow r \stackrel{?}{=} 1 + 2\sin(-\theta)$

$\stackrel{?}{=} 1 - 2\sin\theta$.

$\neq r$

$(-r, \pi - \theta) \Rightarrow -r \stackrel{?}{=} 1 + 2(\sin(\pi - \theta))$.

$-r \stackrel{?}{=} 1 + 2\left[\sin(\pi)\cos(\theta) - \cos(\pi)\sin(\theta)\right]$

$\stackrel{?}{=} 1 + 2\left[0 - -\sin\theta\right]$.

$-r \neq 1 + 2\sin\theta$

not symmetric about x-axis.

b) y-axis.

$$(-r, -\theta) \Rightarrow -r \overset{?}{=} 1 + 2\sin(-\theta).$$

$$-r \neq 1 - 2\sin\theta$$

$$(r, -\theta) \Rightarrow r \overset{?}{=} 1 + 2\sin(\pi - \theta).$$

$$r \overset{?}{=} 1 + 2\left[\sin(\pi)\cos(\theta) - \cos(\pi)\sin(\theta)\right].$$

$$r \overset{?}{=} 1 + 2\left[0 + \sin\theta\right]$$

$$r = 1 + 2\sin\theta.$$

Symmetric about the y-axis.

from a & b ⇒ not symmetric about the origin.

c) about the origin.

$$(-r, \theta) \Rightarrow -r \neq 1 + 2\sin\theta.$$

$$(r, \pi + \theta) \Rightarrow r \overset{?}{=} 1 + 2\sin(\pi + \theta)$$

$$\overset{?}{=} 1 + 2\left[\sin(\pi)\cos(\theta) + \cos(\pi)\sin(\theta)\right].$$

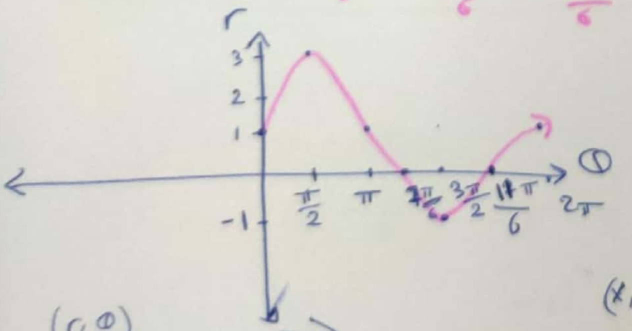
$$\neq 1 - 2\sin\theta.$$

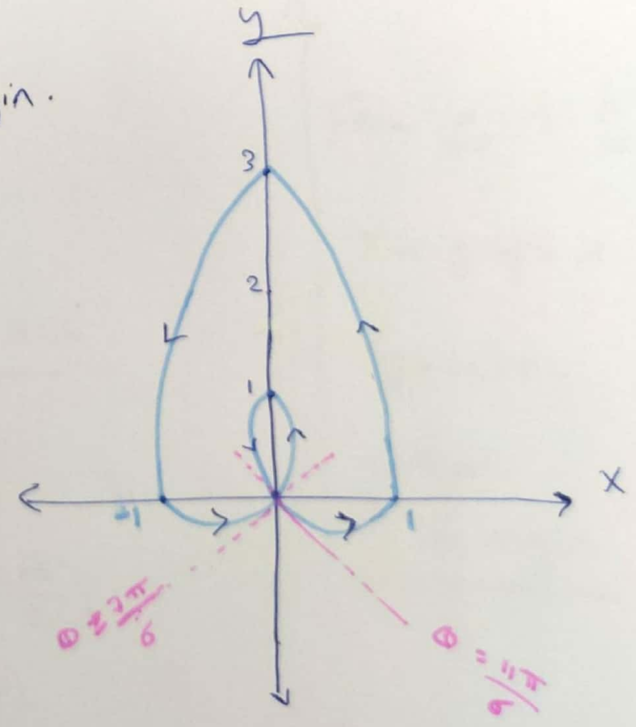
not symmetric about the origin.

$$\Rightarrow r = 1 + 2\sin\theta = 0$$

$$\sin\theta = -\frac{1}{2}$$

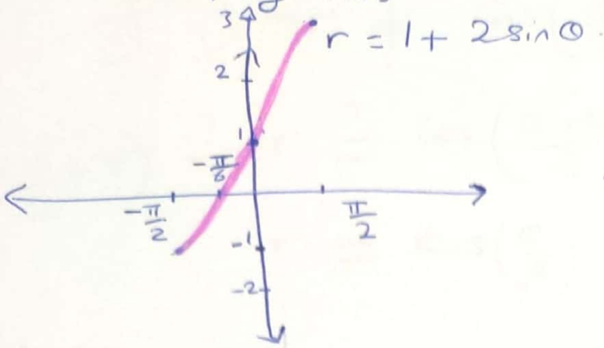
$$\theta = -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \ldots$$

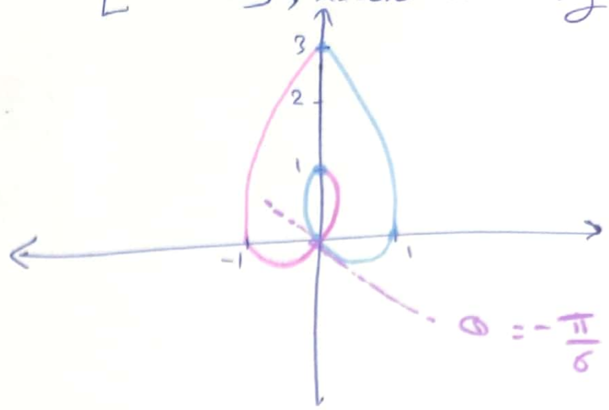




(r, θ)	(x, y)
$(1, 0)$	$(1, 0)$
$(3, \frac{\pi}{2})$	$(0, 3)$
$(1, \pi)$	$(-1, 0)$
$(0, \frac{7\pi}{6})$	$(0, 0)$
$(-1, \frac{3\pi}{2})$	$(0, 1)$

Note You can have a graph on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then use symmetry about y-axis.





(r, θ)	$\Rightarrow$	(x, y)
$(-1, -\frac{\pi}{2})$	$\Rightarrow$	$(0, 1)$
$(0, \frac{-\pi}{6})$	$\Rightarrow$	$(0, 0)$
$(1, 0)$	$\Rightarrow$	$(1, 0)$
$(3, \frac{\pi}{2})$	$\Rightarrow$	$(0, 3)$

From the graph: No symmetry about x-axis or origin.

---

8 $r = \cos\left(\frac{\theta}{2}\right)$

a) x-axis:-

$(r, -\frac{\theta}{2}) \Rightarrow r \stackrel{?}{=} \cos\left(-\frac{\theta}{2}\right)$

$r = \cos\left(-\frac{\theta}{2}\right)$.

The graph is symmetric about x-axis.

b) y-axis:-

$(-r, -\frac{\theta}{2}) \Rightarrow -r \stackrel{?}{=} \cos\left(\frac{-\theta}{2}\right)$

$-r \neq \cos\left(\frac{\theta}{2}\right)$

$(r, \pi - \frac{\theta}{2}) \Rightarrow r \stackrel{?}{=} \cos\left(\frac{2\pi - \theta}{2}\right) = \cos\left(\frac{\theta}{2}\right) = r$

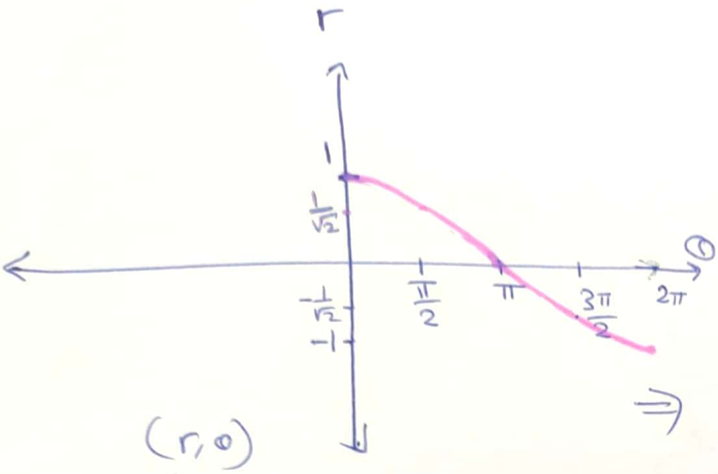
The graph is symmetric about y-axis.

From a & b

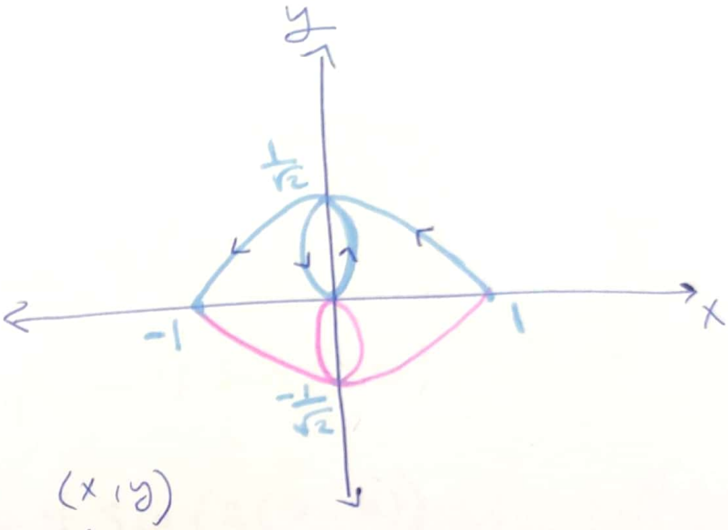
The graph is symmetric about the origin.

$r = \cos\left(\frac{\theta}{2}\right) = 0.$

$\frac{\theta}{2} = \mp\frac{\pi}{2}, \mp\frac{3\pi}{2}, \ldots \Rightarrow \theta = \mp\pi, \mp 3\pi, \ldots$



$\Rightarrow$



(r, θ)	(x, y)
$(1, 0)$	$(1, 0)$
$\left(\frac{1}{\sqrt{2}}, \frac{\pi}{2}\right)$	$\left(0, \frac{1}{\sqrt{2}}\right)$
$(0, \pi)$	$(0, 0)$
$\left(-\frac{1}{\sqrt{2}}, \frac{3\pi}{2}\right)$	$\left(0, \frac{1}{\sqrt{2}}\right)$
$(-1, 2\pi)$	$(-1, 0)$

Note that:

From the graph $\Rightarrow$ The curve has all symmetries.

14 What symmetric does curve have?

$$r^2 = 4\sin(2\theta).$$

x-axis :- $(r, -\theta) \Rightarrow r^2 \overset{?}{=} 4(\sin(-2\theta))$

$$r^2 \neq -4\sin(2\theta)$$

$(-r, \pi - \theta) \Rightarrow (-r)^2 \overset{?}{=} 4\sin(2(\pi-\theta))$

$$r^2 \overset{?}{=} 4\sin(2\pi - 2\theta)$$

$$\overset{?}{=} 4[\sin(2\pi)\cos(2\theta) - \cos(2\pi)\sin(2\theta)]$$

$$\overset{?}{=} 4[0 - \sin(2\theta).$$

$$r^2 \neq -4\sin(2\theta).$$

y axis :- $(-r, -\theta) \Rightarrow (-r)^2 \overset{?}{=} 4(\sin(-2\theta))$

$$r^2 \neq -4\sin(2\theta).$$

$(r, \pi - \theta) \Rightarrow r^2 \overset{?}{=} 4\sin(2(\pi-\theta))$

$$\overset{?}{=} 4\sin(2\pi - 2\theta)$$

$$\overset{?}{=} 4[\sin(2\pi)\cos(2\theta) - \cos(2\pi)\sin(2\theta)]$$

$$r^2 \neq -\sin(2\theta).$$

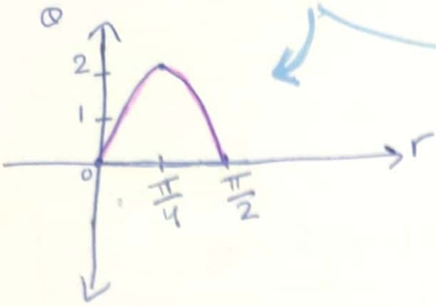
origin:- $(-r, \theta) \Rightarrow (-r)^2 \overset{?}{=} 4\sin(2\theta).$

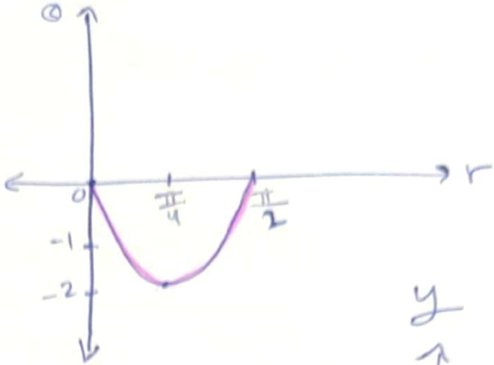
$$r^2 = 4\sin(2\theta)$$

The graph is symmetric about the origin.

$r^2 = 4\sin(2\theta)$.

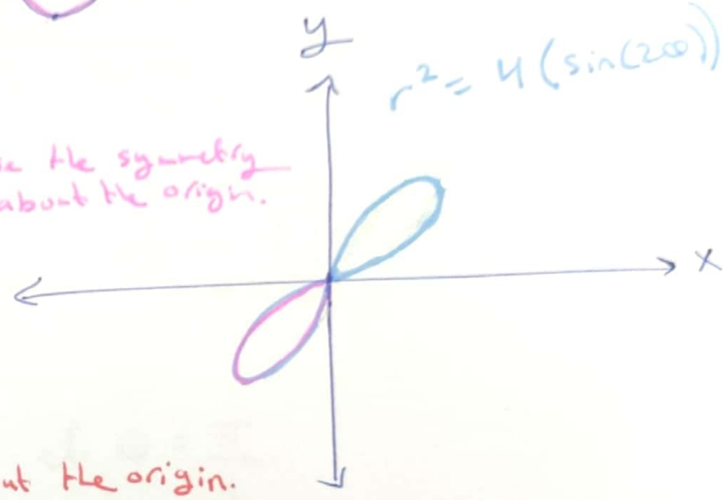
$r = \pm 2\sqrt{\sin 2\theta}$, $\sin(2\theta) \geq 0$, $\sin(2\theta) = 0$
$2\theta = 0, \pi, 2\pi \Rightarrow \theta = 0, \frac{\pi}{2}$





(r, θ)	$\Rightarrow$	(x, y)
$(0, 0)$	$\Rightarrow$	$(0, 0)$
$(2, \frac{\pi}{4})$	$\Rightarrow$	$(\sqrt{2}, \sqrt{2})$
$(0, \frac{\pi}{2})$	$\Rightarrow$	$(0, 0)$

use the symmetry about the origin.



From the graph:
The graph only symmetric about the origin.

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19 Find the slope of the curve.

$r = \sin(2\theta)$, at $\theta = \mp\frac{\pi}{4}$, $\mp\frac{3\pi}{4}$.

$\theta = -\frac{\pi}{4}$, $r = -1$ $(-1, -\frac{\pi}{4})$, $\theta = \frac{\pi}{4}$, $r = 1$ $(1, \frac{\pi}{4})$, $\theta = -\frac{3\pi}{4}$, $r = 1$, $(1, -\frac{3\pi}{4})$, $\theta = \frac{3\pi}{4}$, $r = -1$ $(-1, \frac{3\pi}{4})$

remember:
$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}.$$

$r = f(\theta) = \sin(2\theta)$.

$r' = f'(\theta) = 2\cos(2\theta)$.

$$y' = \frac{2\cos(2\theta)\sin(\theta) + \sin(2\theta)\cos(\theta)}{2\cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta)}$$

$$\boxed{y'\Big|_{\theta = -\frac{\pi}{4}}} = \frac{2\cos(-2\frac{\pi}{4})\sin(-\frac{\pi}{4}) + \sin(-2\frac{\pi}{4})\cos(-\frac{\pi}{4})}{2\cos(-2\frac{\pi}{4})\cos(-\frac{\pi}{4}) - \sin(-\frac{2\pi}{4})\sin(-\frac{\pi}{4})}$$

$$= \frac{-2\cos(\frac{\pi}{2})\sin(\frac{\pi}{4}) - \sin(\frac{\pi}{2})\cos(\frac{\pi}{4})}{2\cos(\frac{\pi}{2})\cos(\frac{\pi}{4}) - \sin(\frac{\pi}{2})\sin(\frac{\pi}{4})}$$

$$y'\Big|_{\theta=-\frac{\pi}{4}} = \frac{0-(1)\left(\frac{1}{\sqrt{2}}\right)}{0-1\left(\frac{1}{\sqrt{2}}\right)} = \boxed{1}$$

= Thes slop at $\theta = -\frac{\pi}{4}$.

$$\boxed{y'\Big|_{\theta=\frac{\pi}{4}}} = \frac{2\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{4}\right)}{2\cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{4}\right)-\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{4}\right)}$$

$$= \frac{0+(1)\frac{1}{\sqrt{2}}}{0-(1)\frac{1}{\sqrt{2}}} = \boxed{-1}$$

= the slope at $\theta = \frac{\pi}{4}$.

$$\boxed{y'\Big|_{\theta=-\frac{3\pi}{4}}} = \frac{2\cos\left(-\frac{3\pi}{2}\right)\sin\left(-\frac{3\pi}{4}\right)+\sin\left(-\frac{3\pi}{2}\right)\cos\left(-\frac{3\pi}{4}\right)}{2\cos\left(-\frac{3\pi}{2}\right)\cos\left(-\frac{3\pi}{4}\right)-\sin\left(-\frac{3\pi}{2}\right)\sin\left(-\frac{3\pi}{4}\right)}$$

$$= \frac{0+-(-1)\left(-\frac{1}{\sqrt{2}}\right)}{0-(-(-1))\left(-\frac{1}{\sqrt{2}}\right)}$$

$$= \frac{-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \boxed{-1}$$

= The slope at $\theta = \frac{-3\pi}{4}$.

$$\left. y' \right|_{\theta=\frac{3\pi}{4}} = \frac{2\cos\left(\frac{3\pi}{2}\right)\sin\left(\frac{3\pi}{4}\right) + \sin\left(\frac{3\pi}{2}\right)\cos\left(\frac{3\pi}{4}\right)}{2\cos\left(\frac{3\pi}{2}\right)\cos\left(\frac{3\pi}{4}\right) - \sin\left(\frac{3\pi}{2}\right)\sin\left(\frac{3\pi}{4}\right)}$$

$$= \frac{0 + (-1)\left(-\frac{1}{\sqrt{2}}\right)}{0 - (-1)\left(\frac{1}{\sqrt{2}}\right)}$$

$$= \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$$

$$= \boxed{1}$$

= The slope at $\theta = \frac{3\pi}{4}$.

Check the symmetry:

x-axis $\Rightarrow (r, -\theta) \Rightarrow r \overset{?}{=} \sin(-2\theta)$

$r \neq 2\sin(2\theta)$

$(-r, \pi-\theta) \Rightarrow -r \overset{?}{=} \sin(2\pi - 2\theta)$

$\overset{?}{=} \sin(2\pi)\cos(2\theta) - \cos(2\pi)\sin(2\theta)$.

$\overset{?}{=} 0 - (1)(\sin(2\theta))$

$-r = -\sin(2\theta)$.

The curve is symmetric about x-axis.

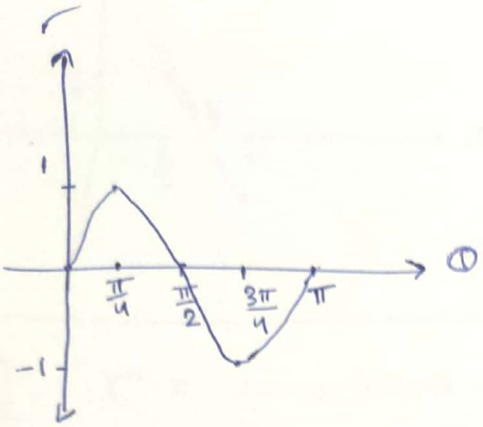
y-axis $\therefore (-r, -\theta) \Rightarrow -r \overset{?}{=} \sin(-2\theta)$.

$-r = -\sin(2\theta)$.

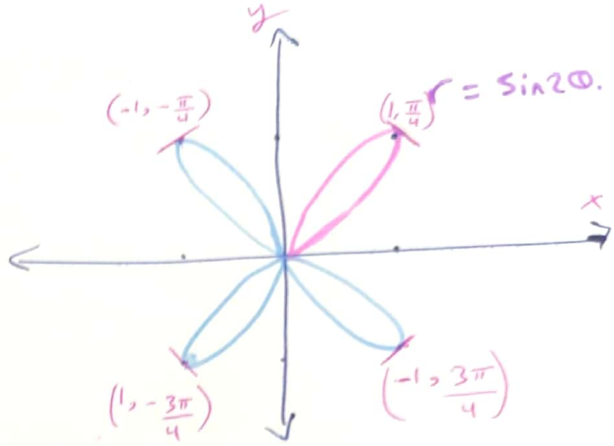
The curve is symmetric about y-axis.

So, the curve is symmetric about the origin.

$r = \sin(2\theta) = 0 \Rightarrow 2\theta = 0, \pi, 2\pi \Rightarrow \theta = 0, \frac{\pi}{2}, \pi.$



$\Rightarrow$



$(r, \theta) \Rightarrow (x, y)$

$(0, 0) \Rightarrow (0, 0)$

$(1, \frac{\pi}{4}) \Rightarrow (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

$(0, \frac{\pi}{2}) \Rightarrow (0, 0)$

Then use the symmetry

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21 Graph:

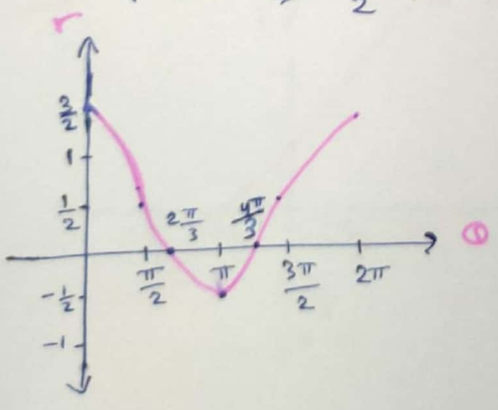
a $r = \frac{1}{2} + \cos\theta.$

The curve is symmetric the x-axis. Check it

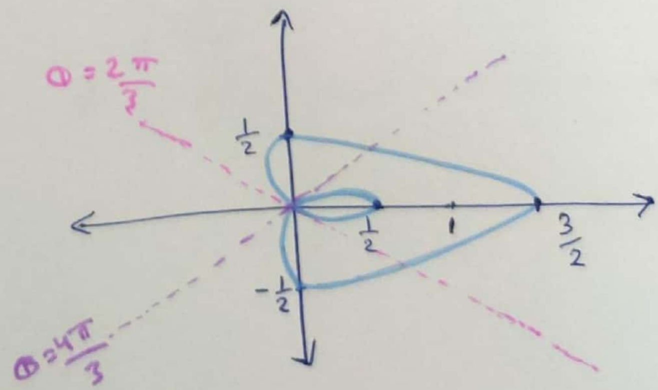
$r = \frac{1}{2} + \cos\theta.$

$r = 0 \Rightarrow \frac{1}{2} + \cos\theta = 0 \Rightarrow \cos\theta = -\frac{1}{2}$

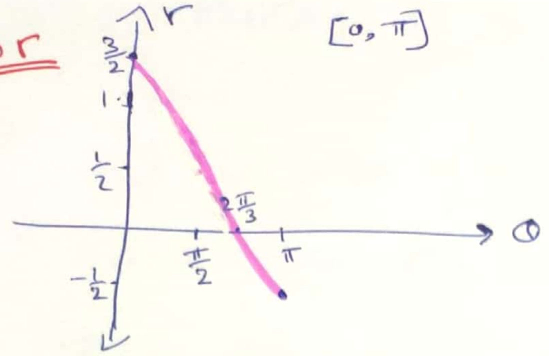
$\Rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}.$



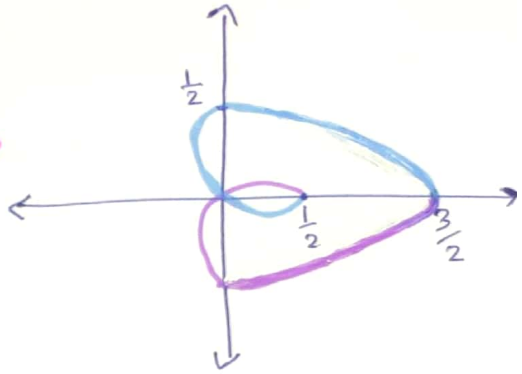
$\Rightarrow$



or



use symmetry about x-axis

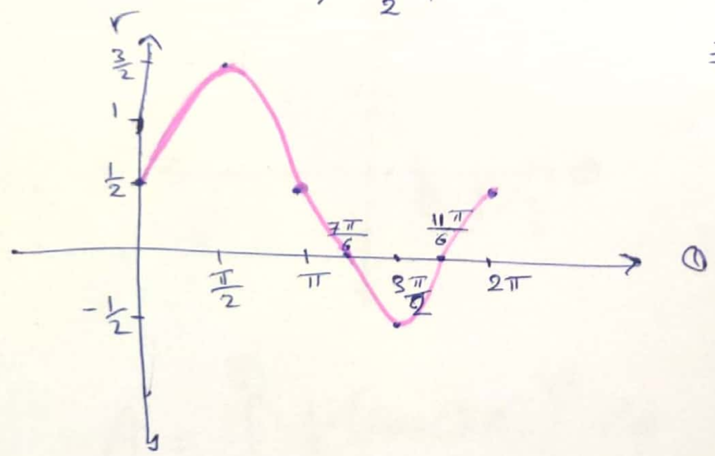


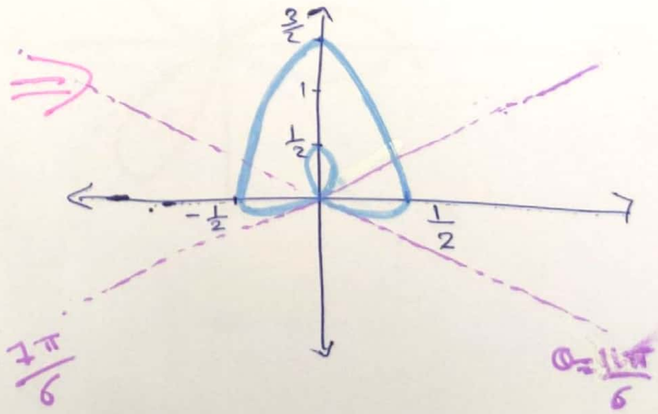
b $r = \frac{1}{2} + \sin\theta$.

The curve is symmetric about y-axis check it

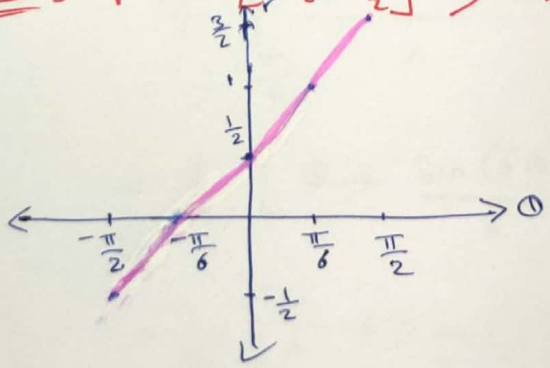
$r = 0 \Rightarrow \frac{1}{2} + \sin\theta = 0 \Rightarrow \sin\theta = -\frac{1}{2}$

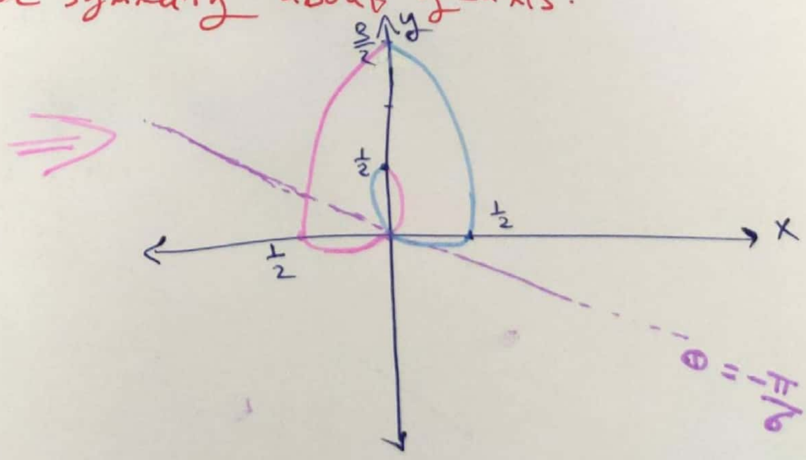
$\Rightarrow \theta = -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$





or graph on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ & use symmetry about y-axis.





11.5 Areas & Lengths in Polar Coordinates.

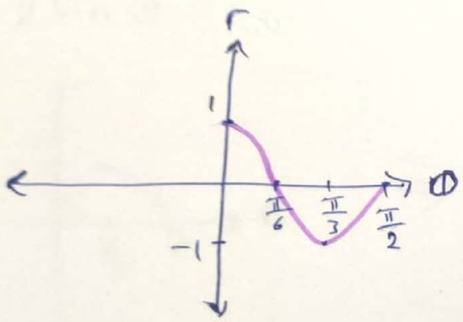
6 Find the area of region inside one leaf of the three-leaved rose $r = \cos(3\theta)$.

* symmetric about x-axis (check it).

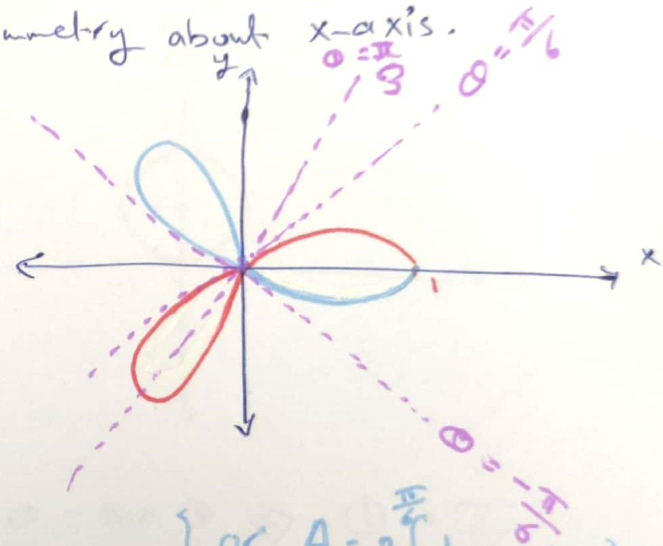
$\cos(3\theta) = 0 \Rightarrow 3\theta = \mp\frac{\pi}{2}, \mp\frac{3\pi}{2}$

$\Rightarrow \theta = \mp\frac{\pi}{6}, \mp\frac{\pi}{2}.$

graph on $[0, \frac{\pi}{2}]$ then use symmetry about x-axis.



$\Longrightarrow$



$$A = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}\left(\cos(3\theta)\right)^2 d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1+\cos(6\theta)}{2}\, d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{\sin(6\theta)}{6}\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$

$$= \frac{1}{4}\left[\frac{\pi}{6} + 0 - \left(-\frac{\pi}{6} + 0\right)\right]$$

$$= \frac{1}{4}\left[\frac{\pi}{6} + \frac{\pi}{6}\right]$$

$$= \boxed{\frac{\pi}{12}}$$

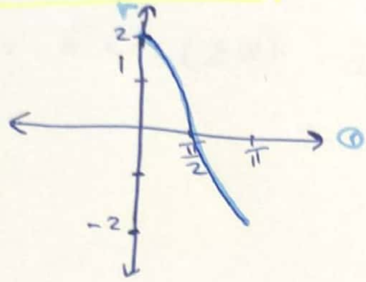
or $A = 2\int_{0}^{\frac{\pi}{6}} \frac{1}{2}\cos^2(3\theta)\, d\theta = \frac{\pi}{12}$

or $A = 2\int_{-\frac{\pi}{6}}^{0} \frac{1}{2}\cos^2(3\theta)\, d\theta$

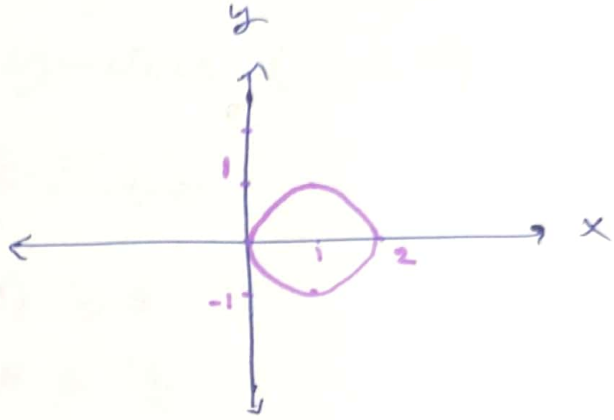
$= \frac{\pi}{12}$

9 Find the area of the region shared by the circles $r = 2\cos\theta$ & $r = 2\sin\theta$.

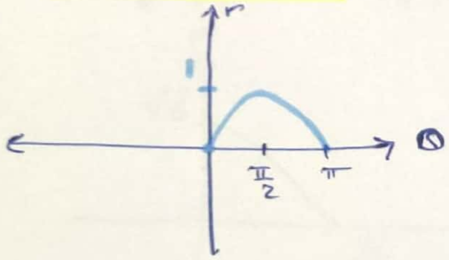
$\boxed{r = 2\cos\theta}$:



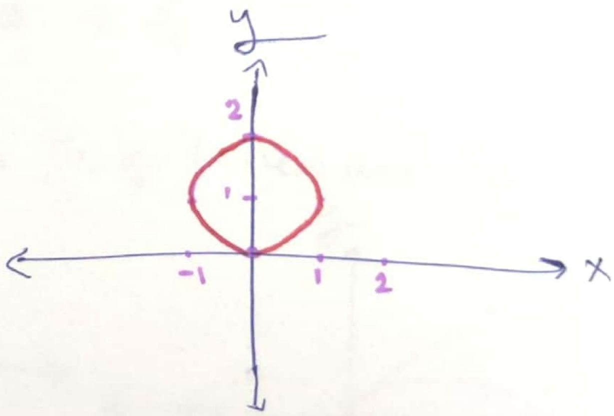
$\Rightarrow$



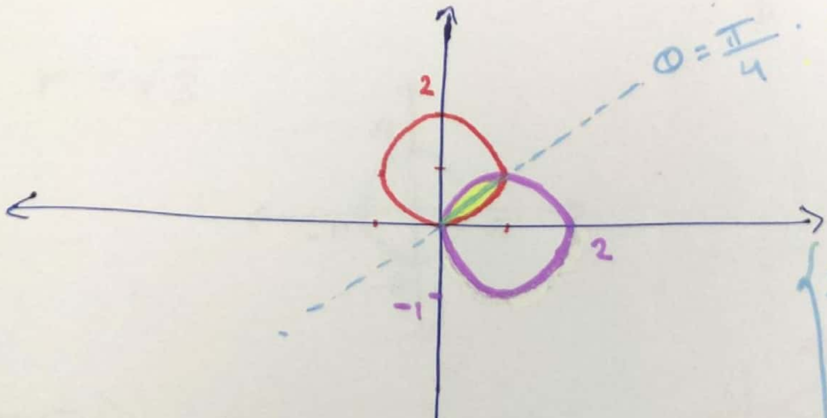
$\boxed{r = 2\sin\theta}$:



$\Rightarrow$



$2\cos\theta = 2\sin\theta \Rightarrow \cos\theta = \sin\theta \Rightarrow \boxed{\theta = \frac{\pi}{4}}$



$$A = 2\int_0^{\frac{\pi}{4}} \frac{1}{2}(2\sin\theta)^2\, d\theta$$

$$= 4\int_0^{\frac{\pi}{4}} \sin^2\theta\, d\theta = 4\int_0^{\frac{\pi}{4}} \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= 2\left[\theta - \frac{\sin(2\theta)}{2}\Big|_0^{\frac{\pi}{4}}\right] = 2\left[\frac{\pi}{4} - \frac{\sin(\frac{\pi}{2})}{2}\right] = \boxed{\frac{\pi}{2} - 1}$$

or $A = 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}(2\cos$

$= \frac{\pi}{2} - 1$

13 Find the area of the region inside the lemniscate $r^2 = 6\cos(2\theta)$ & outside the circle $r = \sqrt{3}$.

$r^2 = 6\cos(2\theta)$ has all symmetries (check it).

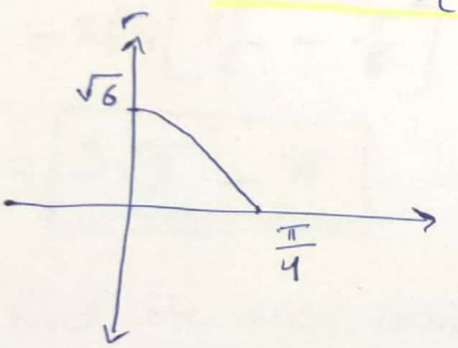
$r^2 = 6\cos(2\theta) \Rightarrow r = \sqrt{6}\sqrt{\cos(2\theta)}$

$\cos(2\theta) \geq 0$

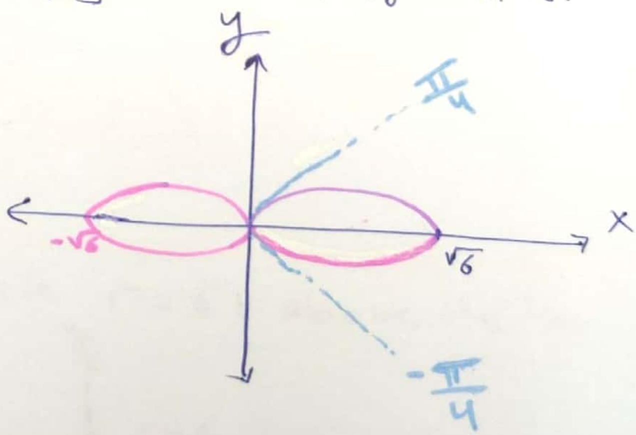
$0 \leq 2\theta \leq \frac{\pi}{2}$.

$0 \leq \theta \leq \frac{\pi}{4}$.

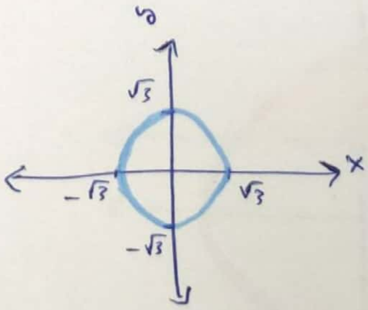
$\Rightarrow$ graph $r = \sqrt{6}\sqrt{\cos(2\theta)}$ on $[0, \frac{\pi}{4}]$ then use symmetries.

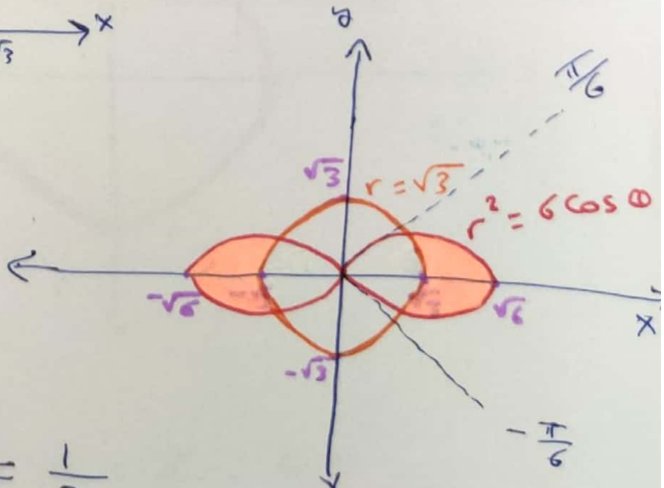


$\longrightarrow$



$\Rightarrow$ graph $r = \sqrt{3}$





$r = r$

$\sqrt{6}\sqrt{\cos(2\theta)} = \sqrt{3}$

$\Rightarrow r^2 = r^2$

$6\cos(2\theta) = 3 \Rightarrow \cos(2\theta) = \frac{1}{2}$

$\Rightarrow 2\theta = \mp\frac{\pi}{3}$.

$\Rightarrow \theta = \mp\frac{\pi}{6}$

$$A = 4\int_0^{\frac{\pi}{6}} \left[\frac{1}{2}\left(\sqrt{6}\sqrt{\cos(2\theta)}\right)^2 - \frac{1}{2}\left(\sqrt{3}\right)^2\right] d\theta.$$

$$= 4\int_0^{\frac{\pi}{6}} \left[\frac{1}{2}\, 6\cos(2\theta) - \frac{3}{2}\right] d\theta$$

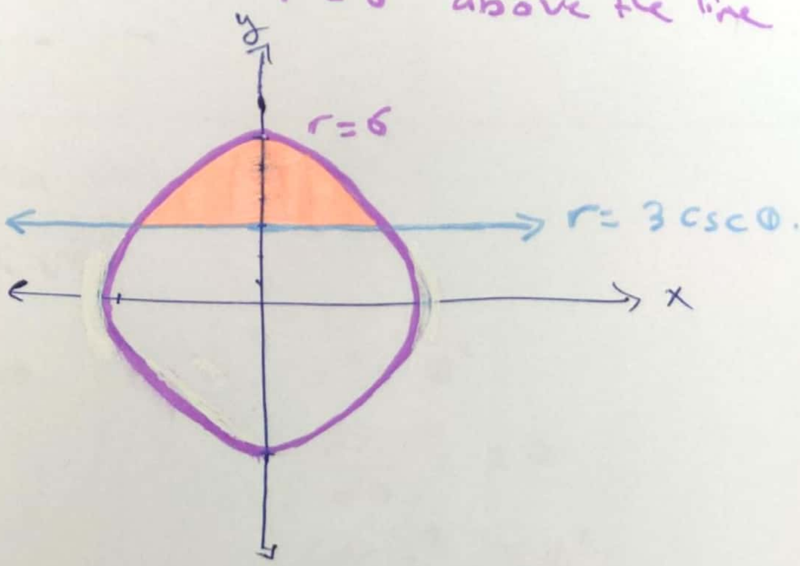
$$= 2\left[\frac{6\sin(2\theta)}{2} - 3\theta \Big|_0^{\frac{\pi}{6}}\right].$$

$$= 2(3)\left[\sin\left(2\cdot\frac{\pi}{6}\right) - \frac{\pi}{6} - (0)\right].$$

$$= 2(3)\left[\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right]$$

$$= \boxed{3\sqrt{3} - \pi}$$

16) Find the area inside the circle $r = 6$ above the line $r = 3\csc\theta$.



$r = 6$

$r = 3\csc\theta$

$r = 3\,\frac{1}{\sin\theta}$

$r\sin\theta = 3$

$y = 3.$

$3\csc\theta = 6$

$\frac{3}{\sin\theta} = 6 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$$A = \int_{\pi/6}^{5\pi/6} \left(\frac{1}{2}(6)^2 - \frac{1}{2}(3\csc\theta)^2\right) d\theta$$

$$= \int_{\pi/6}^{5\pi/6} \left(18 - \frac{9}{2}\csc^2\theta\right) d\theta$$

$$= \left(18\theta + \frac{9}{2}\cot\theta\right)\Bigg|_{\pi/6}^{5\pi/6}$$

$$= 18\frac{(5\pi)}{6} + \frac{9}{2}\cot\left(\frac{5\pi}{6}\right) - 18\frac{\pi}{6} - \frac{9}{2}\cot\left(\frac{\pi}{6}\right)$$

$$= 15\pi + \frac{9}{2}(-\sqrt{3}) - 3\pi - \frac{9}{2}\sqrt{3}$$

$$= \boxed{12\pi - 9\sqrt{3}}$$

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21) Find the length of the curve

$r = \theta^2 \quad , \quad 0 \leq \theta \leq \sqrt{5}.$

$$L = \int_0^{\sqrt{5}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

$r = \theta^2$

$r^2 = \theta^4$

$\frac{dr}{d\theta} = 2\theta$

$\left(\frac{dr}{d\theta}\right)^2 = 4\theta^2.$

$$= \int_0^{\sqrt{5}} \sqrt{\theta^4 + 4\theta^2}\, d\theta.$$

$$= \int_0^{\sqrt{5}} \sqrt{\theta^2}\sqrt{\theta^2+4}\, d\theta = \int_0^{\sqrt{5}} |\theta|\sqrt{\theta^2+4}\, d\theta.$$

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but $|\theta| = \theta$ since $\theta \geq 0$.

So, $\displaystyle\int_0^{\sqrt{5}} \theta\sqrt{\theta^2+4}\, d\theta$.

let $u = \theta^2+4$.

$du = 2\theta\, d\theta$.

$\frac{du}{2} = \theta\, d\theta$

when $\theta = 0 \Rightarrow u = 4$.

$\theta = \sqrt{5} \Rightarrow u = 5+4 = 9$.

$$= \frac{1}{2}\int_4^9 \sqrt{u}\, du.$$

$$= \frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_4^9$$

$$= \frac{1}{3}\left[9^{\frac{3}{2}} - 4^{\frac{3}{2}}\right].$$

$$= \frac{1}{3}\left[\left(9^{\frac{1}{2}}\right)^3 - \left(4^{\frac{1}{2}}\right)^3\right]$$

$$= \frac{1}{3}\left[27 - 8\right)$$

$$= \boxed{\frac{19}{3}}$$

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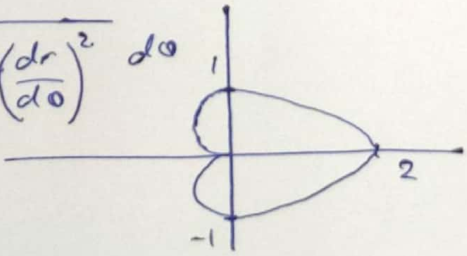
(23) Find the length of the cardioid $r = 1+\cos\theta$.

$$L = \int_0^{2\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta. = 2\int_0^{\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta$$



$$= 2\int_0^{\pi}\sqrt{1+2\cos\theta+\underbrace{\cos^2\theta+\sin^2\theta}_{1}}\, d\theta.$$

$r = 1+\cos\theta$

$r^2 = 1+2\cos\theta+\cos^2\theta$.

$$= 2\int_0^{\pi}\sqrt{2+2\cos\theta}\, d\theta.$$

$\frac{dr}{d\theta} = -\sin\theta$

$\left(\frac{dr}{d\theta}\right)^2 = \sin^2\theta$.

$$= 2\int_0^{\pi} \sqrt{\frac{4}{2}(1+\cos\theta)}\, d\theta.$$

$$= 2\int_0^{\pi} \sqrt{4}\sqrt{\frac{1}{2}+\frac{\cos\theta}{2}}\, d\theta.$$

$$= 4\int_0^{2\pi} \cos\left(\frac{\theta}{2}\right) d\theta.$$

$$= 4 \left.\frac{\sin\left(\frac{\theta}{2}\right)}{\frac{1}{2}}\right|_0^{\pi}$$

$$= 8\left[\sin(\pi) - \sin 0\right]$$

$$= 8[1-0]$$

$$= \boxed{8}$$

$$\cos(2\theta) = 2\cos^2\theta - 1$$

$$\cos(\theta) = 2\cos^2\left(\frac{\theta}{2}\right) - 1$$

$$\frac{\cos\theta + 1}{2} = \cos^2\left(\frac{\theta}{2}\right).$$