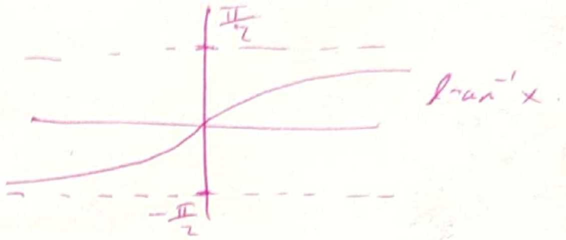
8.7 Improper Integrals.

1) $\displaystyle\int_0^\infty \frac{dx}{x^2+1}$

$= \displaystyle\lim_{b\to\infty} \int_0^b \frac{dx}{x^2+1}$

$= \displaystyle\lim_{b\to\infty} \tan^{-1}x \Big|_0^b$



$= \displaystyle\lim_{b\to\infty} \left(\tan^{-1}b - \tan^{-1}0\right).$

$= \boxed{\frac{\pi}{2}} \rightarrow$ converges

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4) $\displaystyle\int_0^4 \frac{dx}{\sqrt{4-x}}$ $\quad 4-x=0$ if $x=4$.

$= \displaystyle\lim_{b\to 4^-} \int_0^b \frac{dx}{\sqrt{4-x}} = \lim_{b\to 4^-} \left(-2\sqrt{4-x}\Big|_0^b\right)$

$= \displaystyle\lim_{b\to 4^-} \left(-2\sqrt{4-b} - -2\sqrt{4-0}\right).$

$= \displaystyle\lim_{b\to 4^-} \left(-2\sqrt{4-b} + 4\right)$

$= 0 + 4$

$= 4 \rightarrow$ converges.

7) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

$1-x^2=0 \Rightarrow x=1$

$$\lim_{b\to 1^-} \int_0^b \frac{dx}{\sqrt{1-x^2}}$$

$$= \lim_{b\to 1^-} \left(\sin^{-1}(b) - \sin^{-1}(0)\right)$$

$$= \frac{\pi}{2} - 0$$

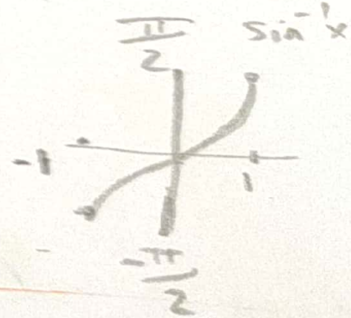
$$= \frac{\pi}{2}.$$

Converges.

remember:

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx.$$

$$= \sin^{-1}\left(\frac{x}{a}\right) + C.$$



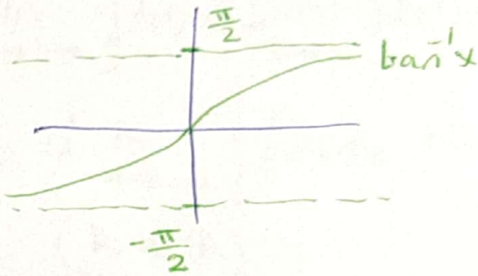
10) $\displaystyle\int_{-\infty}^{2} \frac{2\,dx}{x^2+4}$

remember: $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$

$$= \lim_{a\to-\infty}\left(\int_a^2 \frac{2\,dx}{x^2+4}\right)$$

$$= \lim_{a\to-\infty}\left(2\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\Big|_a^2\right)$$

$$= \lim_{a\to-\infty}\left(\tan^{-1}\left(\frac{2}{2}\right) - \tan^{-1}\left(\frac{a}{2}\right)\right)$$



$$= \frac{\pi}{4} - -\frac{\pi}{2}$$

$$= \frac{\pi}{4} + \frac{2\pi}{4}$$

$$= \boxed{\frac{3\pi}{4}}$$ converges

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14) $\displaystyle\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+4)^{3/2}}$ $\Rightarrow$ let $u = x^2+4$.

$du = 2x\,dx$.

$\frac{du}{2} = x\,dx$

$x = -\infty \Rightarrow u = \infty$

$x = \infty \Rightarrow u = \infty$

$x = 0 \Rightarrow u = 4$.

$$= \int_{-\infty}^{0} \frac{x\,dx}{(x^2+4)^{3/2}} + \int_0^{\infty} \frac{x\,dx}{(x^2+4)^{3/2}}$$

$$= \int_{\infty}^{4} \frac{1}{2(u)^{3/2}}\,du + \int_4^{\infty} \frac{1}{2(u)^{3/2}}\,du$$

$$= \frac{-1}{2}\int_4^{\infty} u^{-3/2}\,du + \frac{1}{2}\int_4^{\infty} u^{-3/2} = 0.$$

$$\Rightarrow -\frac{1}{2}\int_{4}^{\infty} u^{-3/2}\,du + \frac{1}{2}\int_{4}^{\infty} u^{-3/2}\,du$$

$$= \lim_{b\to\infty} -\frac{1}{2}\int_{4}^{b} u^{-3/2}\,du + \lim_{b\to\infty} \frac{1}{2}\int_{4}^{\infty} u^{-3/2}\,du.$$

$$\Rightarrow \lim_{b\to\infty} \frac{1}{2}\int_{4}^{b} u^{-3/2}\,du$$

$$= \lim_{b\to\infty}\left(\frac{1}{2}\cdot u^{-\frac{1}{2}}\cdot -2. \Big|_{4}^{b}\right)$$

$$= \lim_{b\to\infty}\left(-\frac{1}{\sqrt{u}}\Big|_{4}^{b}\right)$$

$$= \lim_{b\to\infty}\left(-\frac{1}{\sqrt{b}} + \frac{1}{2}\right)$$

$$= \frac{1}{2}$$

So, $-\frac{1}{2}\int_{4}^{\infty} u^{-3/2}\,du + \frac{1}{2}\int_{4}^{\infty} u^{-3/2}\,du$

$$= 0$$

$\therefore \int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+4)^{3/2}}$ Converges.

16) $\displaystyle\int_0^2 \frac{s+1}{\sqrt{4-s^2}}\,ds$

$$= \underbrace{\int_0^2 \frac{s}{\sqrt{4-s^2}}\,ds}_{I_1} + \underbrace{\int_0^2 \frac{1}{\sqrt{4-s^2}}\,ds}_{I_2}$$

$$= 2 + \frac{\pi}{2}$$

$$= \frac{4+\pi}{2}$$

$$I_1 = \int_0^2 \frac{s}{\sqrt{4-s^2}}\,ds$$

let $u = 4 - s^2$

$du = -2s\,ds$

$\frac{-du}{2} = s\,ds$

$s=0 \Rightarrow u=4$

$s=2 \Rightarrow u=0$

$$\boxed{I_1} = -\frac{1}{2}\int_4^0 \frac{1}{\sqrt{u}}\,du.$$

$$= \frac{1}{2}\int_0^4 \frac{1}{\sqrt{u}}\,du$$

$$= \lim_{a\to 0^+} \frac{1}{2}\int_a^4 u^{-1/2}\,du.$$

$$= \lim_{a\to 0^+}\left(\frac{1}{2}\cdot 2\sqrt{u}\,\Big|_a^4\right)$$

$$= \lim_{a\to 0^+}\left(2-\sqrt{a}\right)$$

$$= 2-0$$

$$= \boxed{2}$$

$$\boxed{I_2} = \int_0^2 \frac{1}{\sqrt{4-s^2}}\,ds$$

$$= \lim_{b\to 2^-}\int_0^b \frac{1}{\sqrt{4-s^2}}\,ds.$$

$$= \lim_{b\to 2^-}\left(\sin^{-1}\left(\frac{s}{2}\right)\Big|_0^b\right)$$

$$= \lim_{b\to 2^-}\left(\sin^{-1}\left(\frac{b}{2}\right) - \sin^{-1}(0)\right)$$

$$= \frac{\pi}{2} - 0$$

$$= \boxed{\frac{\pi}{2}}$$

(21) $\int_{-\infty}^{0} \theta e^{\theta} \, d\theta$

$$= \lim_{a \to -\infty} \int_{a}^{0} \theta e^{\theta} \, d\theta$$

by parts

$u = \theta \qquad dv = e^{\theta} d\theta$

$du = d\theta \qquad v = e^{\theta}$

$$= \lim_{a \to -\infty} \left[\theta e^{\theta} \Big|_{a}^{0} - \int_{a}^{0} e^{\theta} \, d\theta \right]$$

$$= \lim_{a \to -\infty} \left[0 - a e^{a} - e^{0} + e^{a} \right]$$

$$= \lim_{a \to -\infty} \left[a e^{a} + e^{a} - 1 \right]$$

$$= \left[0 + 0 - 1 \right]$$

$$= \boxed{-1}$$

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(25) $\int_0^1 x \ln x \, dx$

$$= \lim_{a\to 0^+} \int_a^1 x \ln x \, dx$$

by parts: $u = \ln x$ $\quad dv = x\,dx$

$u = \frac{dx}{x}$ $\quad v = \frac{x^2}{2}$

$$= \lim_{a\to 0^+} \left[\ln x \cdot \frac{x^2}{2} \Big|_a^1 - \int_a^1 \frac{x^2}{2} \cdot \frac{dx}{x} \right]$$

$$= \lim_{a\to 0^+} \left[0 - \ln a \cdot \frac{a^2}{2} - \int_a^1 \frac{x}{2} dx \right]$$

$$= \lim_{a\to 0^+} \left[\frac{a^2}{2} \ln a - \frac{x^2}{4} \Big|_a^1 \right].$$

$$= \lim_{a\to 0^+} \left[\frac{a^2}{2} \ln a - \frac{1}{4} + \frac{a^2}{4} \right]$$

$$= 0 - \frac{1}{4} + 0$$

$$= \boxed{-\frac{1}{4}}$$

$\lim_{a\to 0^+} \frac{a^2}{2} \ln a \quad 0 \cdot -\infty$

$= \lim_{a\to 0^+} \frac{\frac{\ln a}{2}}{\frac{1}{a^2}} \quad \frac{-\infty}{\infty}$

$= \lim_{a\to 0^+} \frac{\frac{1}{2a}}{\frac{-2}{a^3}}$

$= \lim_{a\to 0^+} \frac{-a^2}{4} = 0$

(32) $\displaystyle\int_0^2 \frac{dx}{\sqrt{|x-1|}}$

$$= \int_0^1 \frac{dx}{\sqrt{|x-1|}} + \int_1^2 \frac{dx}{\sqrt{|x-1|}}$$

$$= \lim_{b\to 1^-} \int_0^b \frac{dx}{\sqrt{|x-1|}} + \lim_{a\to 1^+} \int_a^2 \frac{dx}{\sqrt{|x-1|}}$$

$|x-1|$

$x-1=0$

$x=1$

$-\ \ +$

$|x-1| = \begin{cases} 1-x & , x<1 \\ x-1 & , x>1 \end{cases}$

$$= \lim_{b\to 1^-} \int_0^b \frac{dx}{\sqrt{1-x}} + \lim_{a\to 1^+} \int_a^2 \frac{dx}{\sqrt{x-1}}$$

$$= \lim_{b\to 1^-} \left[-2\sqrt{1-x}\,\Big|_0^b\right] + \lim_{a\to 1^+} \left[2\sqrt{x-1}\,\Big|_a^2\right].$$

$$= \lim_{b\to 1^-} \left[-2\sqrt{1-b} + 2\sqrt{1}\right] + \lim_{a\to 1^+} \left[2\sqrt{2-1} - 2\sqrt{a-1}\right].$$

$$= 2 + 2$$

$$= \boxed{4.}$$

(40) $\displaystyle\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx.$

let $u = \sqrt{x}$.

$du = \frac{1}{2\sqrt{x}}\, dx.$

$2\,du = \frac{dx}{\sqrt{x}}$

when $x=0 \Rightarrow u=0.$

$x=1 \Rightarrow u=1.$

$$= \int_0^1 2e^{-u}\, du$$

$$= -2e^{-u}\Big|_0^1$$

$$= -2\left[e^{-1} - e^{0}\right]$$

$$= \boxed{\frac{-2}{e} + 2.}$$

37) $\int_0^{\pi} \frac{\sin\theta \, d\theta}{\sqrt{\pi-\theta}}$ Test the Integral for converges.

$$\frac{\sin\theta}{\sqrt{\pi-\theta}} \leq \frac{1}{\sqrt{\pi-\theta}}$$

$\Rightarrow$ by D.C.T.

$$\int_0^{\pi} \frac{1}{\sqrt{\pi-\theta}} \, d\theta = \lim_{b\to\pi^-} \int_0^{b} \frac{1}{\sqrt{\pi-\theta}} \, d\theta$$

$$= \lim_{b\to\pi^-} \left(-2\sqrt{\pi-\theta} \Big|_0^b \right)$$

$$= \lim_{b\to\pi^-} \left(-2\sqrt{\pi-b} + 2\sqrt{\pi} \right)$$

$$= 0 + 2\sqrt{\pi}$$

$$= 2\sqrt{\pi} \quad \text{Converges.}$$

So, $\int_0^{\pi} \frac{\sin\theta}{\sqrt{\pi-\theta}} \, d\theta$ also converges.

41) $\int_0^{\pi} \frac{dt}{\sqrt{t} + \sin t}$, test the integral converges.

remember: when $0 \leq t \leq \pi$

$\Rightarrow 0 \leq \sin t \leq 1$

So, $\Rightarrow \sqrt{t} \leq \sqrt{t} + \sin t \leq 1 + \sqrt{t}$.

$\Rightarrow \frac{1}{\sqrt{t}} \geq \frac{1}{\sqrt{t} + \sin t}$

$\Rightarrow$ Now, we can apply Direct Comparison test (D.C.T).

$\int_0^{\pi} \frac{1}{\sqrt{t}} dt.$

$$= \lim_{a \to 0^+} \int_a^{\pi} \frac{1}{\sqrt{t}} dt = \lim_{a \to 0^+} \left(2\sqrt{t} \Big|_a^{\pi} \right)$$

$$= \lim_{a \to 0^+} \left(2\sqrt{\pi} - 2\sqrt{a} \right)$$

$$= 2\sqrt{\pi} - 0$$

$$= 2\sqrt{\pi}$$

converges.

So, by D.C.T $\int_0^{\pi} \frac{dt}{\sqrt{t} + \sin t}$ converges.

47) $\int_1^\infty \frac{dx}{x^3+1}$ (Test the integral converges??)

$x^3+1 \geq x^3$ $(x\geq 1)$

$\frac{1}{x^3+1} \leq \frac{1}{x^3}$

by direct comparison tests (D.C.T).

$$0 \leq \frac{1}{x^3+1} \leq \frac{1}{x^3} \quad \text{for } x \geq 1.$$

$$\int_1^\infty \frac{1}{x^3}\, dx.$$

$$= \lim_{b\to\infty} \int_1^b \frac{1}{x^3}\, dx.$$

$$= \lim_{b\to\infty} \left(\frac{1}{-2x^2} \Big|_1^b \right) = \lim_{b\to\infty} \left[\frac{1}{-2b^2} + \frac{1}{2} \right] = \boxed{\frac{1}{2}} \text{ converges.}$$

So, $\int_1^\infty \frac{dx}{x^3+1}$ also converges.

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50) Test the integrals converges.

$\int_0^\infty \frac{d\theta}{1+e^\theta}$

$1+e^\theta \geq e^\theta \quad \theta \geq 0$

$\frac{1}{1+e^\theta} \leq \frac{1}{e^\theta}$

$0 \leq \frac{1}{1+e^\theta} \leq \frac{1}{e^\theta}$, by D.C.T :-

$$\int_0^\infty \frac{1}{e^\theta}\, d\theta = \lim_{b\to\infty} \int_0^b e^{-\theta}\, d\theta.$$

$$= \lim_{b\to\infty} \left(-e^{-\theta} \Big|_0^b \right).$$

$$= \lim_{b\to\infty} \left(-e^{-b} - -e^{0} \right)$$

$$= 0 + 1$$

$$= 1 \text{ converges.}$$

$\therefore \int_0^\infty \frac{d\theta}{1+e^\theta}$ converges.

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(58) $\int_2^\infty \frac{1}{\ln x}\, dx$. Test the integral converges.

we know that $x > \ln x$

$\Rightarrow \frac{1}{x} < \frac{1}{\ln x}$.

$$\int_2^\infty \frac{1}{x}\, dx = \lim_{b\to\infty}\left(\int_2^b \frac{1}{x}\, dx\right)$$

$$= \lim_{b\to\infty}\left(\ln|x| \Big|_2^b\right)$$

$$= \lim_{b\to\infty}\left(\ln|b| - \ln|2|\right)$$

$$= \infty \quad \text{(diverges)}.$$

So, by direct comparison test:

$\int_2^\infty \frac{1}{\ln x}\, dx$ diverges.

56) $\displaystyle\int_{\pi}^{\infty} \frac{1+\sin x}{x^2}\, dx$.

$\sin x \leq 1$

$1+\sin x \leq 2$

$\frac{1+\sin x}{x^2} \leq \frac{2}{x^2}$

$$\frac{1+\sin x}{x^2} \leq \frac{2}{x^2}$$

by D.C.T.

$$\int_{\pi}^{\infty} \frac{2}{x^2}\, dx = \lim_{b\to\infty} \int_{\pi}^{b} \frac{2}{x^2}\, dx = \lim_{b\to\infty} \left(\frac{-2}{x} \Big|_{\pi}^{b} \right)$$

$$= \lim_{b\to\infty} \left(-\frac{2}{b} - \frac{-2}{\pi} \right)$$

$$= \boxed{\frac{2}{\pi}}$$ Converges.

So, $\displaystyle\int_{\pi}^{\infty} \frac{1+\sin x}{x^2}\, dx$ also converges.

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62) $\displaystyle\int_{1}^{\infty} \frac{1}{e^x - 2^x}\, dx$. (apply Limit Comparison Test)

let $f(x) = \frac{1}{e^x - 2^x}$ & $g(x) = \frac{1}{e^x}$

$$\Rightarrow \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\frac{1}{e^x - 2^x}}{\frac{1}{e^x}} = \lim_{x\to\infty} \frac{e^x}{e^x - 2^x}$$

$$\lim_{x\to\infty} \frac{e^x}{e^x - 2^x} = \lim_{x\to\infty} \frac{\frac{e^x}{e^x}}{\frac{e^x - 2^x}{e^x}} = \lim_{x\to\infty} \frac{1}{1 - \left(\frac{2}{e}\right)^x} = \frac{1}{1-0} = \boxed{1}$$

$$\& \int_{1}^{\infty} \frac{1}{e^x} = \lim_{b\to\infty} \int_{1}^{b} \frac{1}{e^x}\, dx = \lim_{b\to\infty} \left[-e^{-x} \Big|_{1}^{b} \right] = \lim_{b\to\infty} \left[-e^{-b} + \frac{1}{e} \right]$$

$$= 0 + \frac{1}{e} = \boxed{\frac{1}{e}}$$

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$$\Rightarrow \int_1^\infty \frac{1}{e^x}\,dx \text{ converges} \Rightarrow \int_1^\infty \frac{1}{e^x - 2^x}\,dx \text{ converges}$$

(by Limit Comparison Test).

(65) Find the values of P for which each integral converges.

a) $\int_1^2 \frac{dx}{x(\ln x)^p}$, b) $\int_2^\infty \frac{dx}{x(\ln x)^p}$

let $u = \ln x$, when $x = 1 \Rightarrow u = 0$.

$du = \frac{dx}{x}$, $x = 2 \Rightarrow u = \ln 2$.

$x = \infty \Rightarrow u = \infty$

a) $\int_0^{\ln 2} \frac{du}{u^p}$, b) $\int_{\ln 2}^\infty \frac{du}{u^p}$

a) $\int_0^{\ln 2} \frac{du}{u^p}$

for $p = 1 \Rightarrow \int_0^{\ln 2} \frac{du}{u} = \lim_{a \to 0^+} \int_a^{\ln 2} \frac{du}{u} = \lim_{a \to 0^+} \ln|u| \Big|_a^{\ln 2}$

$= \lim_{a \to 0^+} (\ln \ln 2 - \ln a)$

$\boxed{= \infty}$

for $p \neq 1 \Rightarrow \int_0^{\ln 2} \frac{du}{u^p} = \lim_{a \to 0^+} \int_a^{\ln 2} \frac{du}{u^p} = \lim_{a \to 0^+} \frac{u^{-p+1}}{-p+1} \Big|_a^{\ln 2}$

$= \lim_{a \to 0^+} \left(\frac{(\ln 2)^{1-p}}{1-p} - \frac{a^{1-p}}{1-p} \right)$

$= \begin{cases} \frac{(\ln 2)^{1-p}}{1-p}, & p < 1 \\ \infty, & p > 1 \end{cases}$

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$$\boxed{\text{So, } \int_1^2 \frac{dx}{x(\ln x)^p} = \begin{cases} \dfrac{(\ln 2)^{1-p}}{1-p} & , p<1 \\ \infty & , p \geq 1 \end{cases}}$$

Converges for $p<1$.

diverges for $p \geq 1$.

b) $\displaystyle\int_{\ln 2}^{\infty} \frac{du}{u^p}$

for $p=1 \Rightarrow \displaystyle\int_{\ln 2}^{\infty} \frac{du}{u} = \lim_{b\to\infty} \int_{\ln 2}^{b} \frac{du}{u} = \lim_{b\to\infty} \ln|u| \Big|_{\ln 2}^{b}$

$$= \lim_{b\to\infty} \ln b - \ln 2$$

$$= \infty .$$

for $p \neq 1 \Rightarrow \displaystyle\int_{\ln 2}^{\infty} \frac{du}{u^p} = \lim_{b\to\infty} \int_{\ln 2}^{b} \frac{du}{u^p}$

$$= \lim_{b\to\infty} \frac{u^{-p+1}}{-p+1} \Bigg|_{\ln 2}^{b}$$

$$= \lim_{b\to\infty} \frac{b^{-p+1}}{-p+1} - \frac{\ln 2^{-p+1}}{-p+1}$$

$$= \begin{cases} \dfrac{(\ln 2)^{1-p}}{p-1} & , p>1 \\ \infty & , p<1 \end{cases}$$

$$\boxed{\text{So, } \int_2^{\infty} \frac{dx}{x(\ln x)^p} = \begin{cases} \dfrac{(\ln 2)^{1-p}}{p-1} & , p>1 \\ \infty & , p \leq 1. \end{cases}}$$

converges for $p>1$.

diverges for $p \leq 1$.