8.7 Improper Integrals

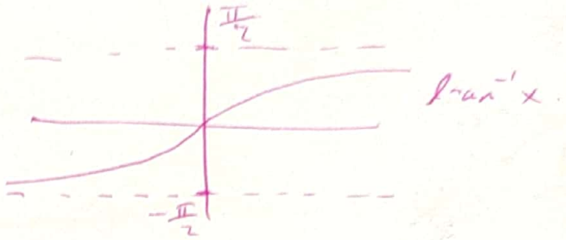
1) $\displaystyle\int_0^{\infty} \frac{dx}{x^2+1}$

$\displaystyle = \lim_{b\to\infty} \int_0^{b} \frac{dx}{x^2+1}$

$\displaystyle = \lim_{b\to\infty} \tan^{-1}x \Big|_0^b$

$\displaystyle = \lim_{b\to\infty} \left(\tan^{-1}b - \tan^{-1}0\right).$

$\displaystyle = \boxed{\frac{\pi}{2}} \rightarrow$ converges



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4) $\displaystyle\int_0^{4} \frac{dx}{\sqrt{4-x}}$ $4-x=0$ if $x=4.$

$\displaystyle = \lim_{b\to 4^-} \int_0^{b} \frac{dx}{\sqrt{4-x}} = \lim_{b\to 4^-} \left(-2\sqrt{4-x}\Big|_0^b\right)$

$\displaystyle = \lim_{b\to 4^-} \left(-2\sqrt{4-b} - -2\sqrt{4-0}\right).$

$\displaystyle = \lim_{b\to 4^-} \left(-2\sqrt{4-b} + 4\right)$

$= 0 + 4$

$= 4 \rightarrow$ converges.

7) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

$1 - x^2 = 0 \Rightarrow \underline{x = 1}$

$$\lim_{b \to 1^-} \int_0^b \frac{dx}{\sqrt{1-x^2}}$$

$$= \lim_{b \to 1^-} \left(\sin^{-1}(b) - \sin^{-1}(0) \right)$$

$$= \frac{\pi}{2} - 0$$

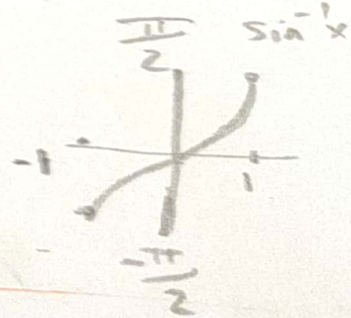
$$= \frac{\pi}{2}.$$

Converges.

remember:

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx$$

$$= \sin^{-1}\left(\frac{x}{a}\right) + C.$$



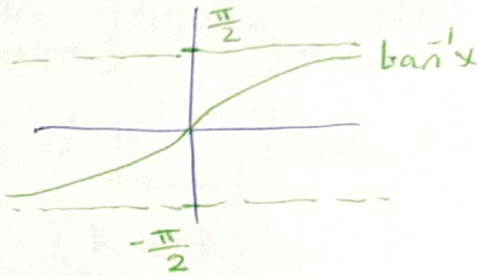
10) $\displaystyle\int_{-\infty}^{2} \frac{2\,dx}{x^2+4}$

remember: $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$

$$= \lim_{a\to-\infty}\left(\int_a^2 \frac{2\,dx}{x^2+4}\right)$$

$$= \lim_{a\to-\infty}\left(2\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\Big|_a^2\right)$$

$$= \lim_{a\to-\infty}\left(\tan^{-1}\left(\frac{2}{2}\right) - \tan^{-1}\left(\frac{a}{2}\right)\right)$$



$$= \frac{\pi}{4} - -\frac{\pi}{2}$$

$$= \frac{\pi}{4} + \frac{2\pi}{4}$$

$$= \boxed{\frac{3\pi}{4}}$$ converges

14) $\displaystyle\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+4)^{3/2}}$

$\Rightarrow$ let $u = x^2+4$.

$du = 2x\,dx$.

$\frac{du}{2} = x\,dx$

$x = -\infty \Rightarrow u = \infty$

$x = \infty \Rightarrow u = \infty$

$x = 0 \Rightarrow u = 4$.

$$= \int_{-\infty}^{0} \frac{x\,dx}{(x^2+4)^{3/2}} + \int_0^{\infty} \frac{x\,dx}{(x^2+4)^{3/2}}$$

$$= \int_{\infty}^{4} \frac{1}{2(u)^{3/2}}\,du + \int_4^{\infty} \frac{1}{2(u)^{3/2}}\,du$$

$$= \frac{-1}{2}\int_4^{\infty} u^{-3/2}\,du + \frac{1}{2}\int_4^{\infty} u^{-3/2} = 0.$$

$$\Rightarrow -\frac{1}{2}\int_{4}^{\infty} u^{-3/2}\,du + \frac{1}{2}\int_{4}^{\infty} u^{-3/2}\,du$$

$$= \lim_{b\to\infty} -\frac{1}{2}\int_{4}^{b} u^{-3/2}\,du + \lim_{b\to\infty} \frac{1}{2}\int_{4}^{b} u^{-3/2}\,du.$$

$$\Rightarrow \lim_{b\to\infty} \frac{1}{2}\int_{4}^{b} u^{-3/2}\,du$$

$$= \lim_{b\to\infty}\left(\frac{1}{2}\cdot u^{-\frac{1}{2}}\cdot -2.\Big|_{4}^{b}\right)$$

$$= \lim_{b\to\infty}\left(-\frac{1}{\sqrt{u}}\Big|_{4}^{b}\right)$$

$$= \lim_{b\to\infty}\left(-\frac{1}{\sqrt{b}} + \frac{1}{2}\right)$$

$$= \frac{1}{2}$$

So, $-\frac{1}{2}\int_{4}^{\infty} u^{-3/2}\,du + \frac{1}{2}\int_{4}^{\infty} u^{-3/2}\,du$

$$= 0$$

$\therefore \int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+4)^{3/2}}$ Converges.

16) $\int_0^2 \frac{s+1}{\sqrt{4-s^2}}\, ds$

$$= \underbrace{\int_0^2 \frac{s}{\sqrt{4-s^2}}\, ds}_{I_1} + \underbrace{\int_0^2 \frac{1}{\sqrt{4-s^2}}\, ds}_{I_2}$$

$$= 2 + \frac{\pi}{2}$$

$$= \frac{4+\pi}{2}$$

$I_1 = \int_0^2 \frac{s}{\sqrt{4-s^2}}\, ds$

let $u = 4 - s^2$

$du = -2s\, ds$

$-\frac{du}{2} = s\, ds$

$s = 0 \Rightarrow u = 4$

$s = 2 \Rightarrow u = 0$

$$\boxed{I_1} = -\frac{1}{2}\int_4^0 \frac{1}{\sqrt{u}}\, du.$$

$$= \frac{1}{2}\int_0^4 \frac{1}{\sqrt{u}}\, du$$

$$= \lim_{a\to 0^+} \frac{1}{2}\int_a^4 u^{-1/2}\, du.$$

$$= \lim_{a\to 0^+}\left(\frac{1}{2}\cdot 2\sqrt{u}\,\Big|_a^4\right)$$

$$= \lim_{a\to 0^+}\left(2-\sqrt{a}\right)$$

$$= 2 - 0$$

$$= \boxed{2}$$

$$\boxed{I_2} = \int_0^2 \frac{1}{\sqrt{4-s^2}}\, ds$$

$$= \lim_{b\to 2^-}\int_0^b \frac{1}{\sqrt{4-s^2}}\, ds.$$

$$= \lim_{b\to 2^-}\left(\sin^{-1}\left(\frac{s}{2}\right)\Big|_0^b\right)$$

$$= \lim_{b\to 2^-}\left(\sin^{-1}\left(\frac{b}{2}\right) - \sin^{-1}(0)\right)$$

$$= \frac{\pi}{2} - 0$$

$$= \boxed{\frac{\pi}{2}}$$

(21) $\int_{-\infty}^{0} \theta e^{\theta} \, d\theta$

$= \lim_{a \to -\infty} \int_{a}^{0} \theta e^{\theta} \, d\theta$

by parts

$u = \theta \qquad dv = e^{\theta} d\theta$

$du = d\theta \qquad v = e^{\theta}$

$= \lim_{a \to -\infty} \left[\theta e^{\theta} \Big|_{a}^{0} - \int_{a}^{0} e^{\theta} \, d\theta \right]$

$= \lim_{a \to -\infty} \left[0 - a e^{a} - e^{0} + e^{a} \right]$

$= \lim_{a \to -\infty} \left[a e^{a} + e^{a} - 1 \right]$

$= \left[0 + 0 - 1 \right]$

$= \boxed{-1}$

(25) $\int_0^1 x \ln x \, dx$

$= \lim_{a \to 0^+} \int_a^1 x \ln x \, dx$

by parts: $u = \ln x \qquad dv = x\,dx$

$u = \frac{dx}{x} \qquad v = \frac{x^2}{2}$

$= \lim_{a \to 0^+} \left[\ln x \cdot \frac{x^2}{2} \Big|_a^1 - \int_a^1 \frac{x^2}{2} \cdot \frac{dx}{x} \right]$

$= \lim_{a \to 0^+} \left[0 - \ln a \cdot \frac{a^2}{2} - \int_a^1 \frac{x}{2} \, dx \right]$

$= \lim_{a \to 0^+} \left[\frac{a^2}{2} \ln a - \frac{x^2}{4} \Big|_a^1 \right]$

$= \lim_{a \to 0^+} \left[\frac{a^2}{2} \ln a - \frac{1}{4} + \frac{a^2}{4} \right]$

$= 0 - \frac{1}{4} + 0$

$= \boxed{-\frac{1}{4}}$

$\lim_{a \to 0^+} \frac{a^2}{2} \ln a \qquad 0 \cdot -\infty$

$= \lim_{a \to 0^+} \frac{\frac{\ln a}{2}}{\frac{1}{a^2}} \qquad \frac{-\infty}{\infty}$

$= \lim_{a \to 0^+} \frac{\frac{1}{2a}}{\frac{-2}{a^3}}$

$= \lim_{a \to 0^+} \frac{-a^2}{4} = 0$

(32) $\displaystyle\int_0^2 \frac{dx}{\sqrt{|x-1|}}$

$$= \int_0^1 \frac{dx}{\sqrt{|x-1|}} + \int_1^2 \frac{dx}{\sqrt{|x-1|}}$$

$$= \lim_{b\to 1^-} \int_0^b \frac{dx}{\sqrt{|x-1|}} + \lim_{a\to 1^+} \int_a^2 \frac{dx}{\sqrt{|x-1|}}$$

$|x-1|$

$x-1=0$

$x=1$

$-\quad +$

$$|x-1| = \begin{cases} 1-x & , x<1 \\ x-1 & , x>1 \end{cases}$$

$$= \lim_{b\to 1^-} \int_0^b \frac{dx}{\sqrt{1-x}} + \lim_{a\to 1^+} \int_a^2 \frac{dx}{\sqrt{x-1}}$$

$$= \lim_{b\to 1^-} \left[-2\sqrt{1-x} \Big|_0^b \right] + \lim_{a\to 1^+} \left[2\sqrt{x-1} \Big|_a^2 \right].$$

$$= \lim_{b\to 1^-} \left[-2\sqrt{1-b} + 2\sqrt{1} \right] + \lim_{a\to 1^+} \left[2\sqrt{2-1} - 2\sqrt{a-1} \right].$$

$$= 2 + 2$$

$$= \boxed{4.}$$

(40) $\displaystyle\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx.$

let $u = \sqrt{x}$.

$du = \frac{1}{2\sqrt{x}}\, dx.$

$2\,du = \frac{dx}{\sqrt{x}}$

when $x=0 \Rightarrow u=0.$

$x=1 \Rightarrow u=1.$

$$= \int_0^1 2e^{-u}\, du$$

$$= -2e^{-u} \Big|_0^1$$

$$= -2\left[e^{-1} - e^{0}\right]$$

$$= \boxed{\frac{-2}{e} + 2.}$$

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37) $\int_0^{\pi} \frac{\sin\theta \, d\theta}{\sqrt{\pi - \theta}}$ Test the Integral for converges.

$$\frac{\sin\theta}{\sqrt{\pi-\theta}} \leq \frac{1}{\sqrt{\pi-\theta}}.$$

$\Rightarrow$ by D.C.T.

$$\int_0^{\pi} \frac{1}{\sqrt{\pi-\theta}} \, d\theta = \lim_{b \to \pi^-} \int_0^{b} \frac{1}{\sqrt{\pi-\theta}} \, d\theta.$$

$$= \lim_{b \to \pi^-} \left(-2\sqrt{\pi-\theta} \Big|_0^b \right)$$

$$= \lim_{b \to \pi^-} \left(-2\sqrt{\pi - b} + 2\sqrt{\pi} \right).$$

$$= 0 + 2\sqrt{\pi}$$

$$= 2\sqrt{\pi}$$ Converges.

So, $\int_0^{\pi} \frac{\sin\theta}{\sqrt{\pi-\theta}} \, d\theta$ also converges.

41) $\int_0^{\pi} \frac{dt}{\sqrt{t} + \sin t}$, test the integral converges.

remember: when $0 \leq t \leq \pi$

$\Rightarrow 0 \leq \sin t \leq 1$

So, $\Rightarrow \sqrt{t} \leq \sqrt{t} + \sin t \leq 1 + \sqrt{t}$.

$\Rightarrow \frac{1}{\sqrt{t}} \geq \frac{1}{\sqrt{t} + \sin t}$

$\Rightarrow$ Now, we can apply Direct Comparison test (D.C.T).

$\int_0^{\pi} \frac{1}{\sqrt{t}} \, dt$.

$= \lim_{a \to 0^+} \int_a^{\pi} \frac{1}{\sqrt{t}} \, dt = \lim_{a \to 0^+} \left(2\sqrt{t} \Big|_a^{\pi} \right)$

$= \lim_{a \to 0^+} \left(2\sqrt{\pi} - 2\sqrt{a} \right)$

$= 2\sqrt{\pi} - 0$

$= 2\sqrt{\pi}$

converges.

So, by D.C.T $\int_0^{\pi} \frac{dt}{\sqrt{t} + \sin t}$ converges.

47) $\int_1^\infty \frac{dx}{x^3+1}$ (Test the integral converges??)

$x^3+1 \geq x^3$ $(x\geq 1)$

$\frac{1}{x^3+1} \leq \frac{1}{x^3}$

by direct comparison tesby (D.C.T).

$$0 \leq \frac{1}{x^3+1} \leq \frac{1}{x^3} \quad \text{for } x\geq 1.$$

$$\int_1^\infty \frac{1}{x^3}\,dx.$$

$$= \lim_{b\to\infty} \int_1^b \frac{1}{x^3}\,dx.$$

$$= \lim_{b\to\infty}\left(\frac{1}{-2x^2}\Big|_1^b\right) = \lim_{b\to\infty}\left[\frac{1}{-2b^2}+\frac{1}{2}\right] = \boxed{\frac{1}{2}} \text{ converges}$$

So, $\int_1^\infty \frac{dx}{x^3+1}$ also converges.

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50) Test the integrals converges.

$\int_0^\infty \frac{d\theta}{1+e^\theta}$

$1+e^\theta \geq e^\theta$ $\theta\geq 0$

$\frac{1}{1+e^\theta} \leq \frac{1}{e^\theta}$

$0 \leq \frac{1}{1+e^\theta} \leq \frac{1}{e^\theta}$, by D.C.T :>

$$\int_0^\infty \frac{1}{e^\theta}\,d\theta = \lim_{b\to\infty}\int_0^b e^{-\theta}\,d\theta.$$

$$= \lim_{b\to\infty}\left(-e^{-\theta}\Big|_0^b\right).$$

$$= \lim_{b\to\infty}\left(-e^{-b} - -e^{0}\right)$$

$$= 0+1$$

$= 1$ converges.

$\therefore \int_0^\infty \frac{d\theta}{1+e^\theta}$ converges.

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58 $\int_2^{\infty} \frac{1}{\ln x} dx$. Test the integral converges.

we know that $x > \ln x$

$\Rightarrow \frac{1}{x} < \frac{1}{\ln x}$.

$$\int_2^{\infty} \frac{1}{x} dx = \lim_{b \to \infty} \left(\int_2^{b} \frac{1}{x} dx \right)$$

$$= \lim_{b \to \infty} \left(\ln|x| \Big|_2^{b} \right)$$

$$= \lim_{b \to \infty} \left(\ln|b| - \ln|2| \right)$$

$$= \infty \quad \text{(diverges)}.$$

So, by direct comparison test:

$\int_2^{\infty} \frac{1}{\ln x} dx$ diverges.

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(56) $\displaystyle\int_{\pi}^{\infty} \frac{1+\sin x}{x^2}\,dx.$

$\sin x \leq 1$

$1+\sin x \leq 2$

$\frac{1+\sin x}{x^2} \leq \frac{2}{x^2}$

$$\frac{1+\sin x}{x^2} \leq \frac{2}{x^2}$$

by D.C.T.

$$\int_{\pi}^{\infty} \frac{2}{x^2}\,dx. = \lim_{b\to\infty} \int_{\pi}^{b} \frac{2}{x^2}\,dx = \lim_{b\to\infty} \left(\frac{-2}{x} \Big|_{\pi}^{b} \right)$$

$$= \lim_{b\to\infty} \left(-\frac{2}{b} - \frac{-2}{\pi} \right)$$

$$= \boxed{\frac{2}{\pi}}$$ Converges.

So, $\displaystyle\int_{\pi}^{\infty} \frac{1+\sin x}{x^2}\,dx$ also converges.

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(62) $\displaystyle\int_{1}^{\infty} \frac{1}{e^x - 2^x}\,dx.$

(apply Limit Comparison Test)

let $f(x) = \dfrac{1}{e^x - 2^x}$ & $g(x) = \dfrac{1}{e^x}$

$$\Rightarrow \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\frac{1}{e^x - 2^x}}{\frac{1}{e^x}} = \lim_{x\to\infty} \frac{e^x}{e^x - 2^x}$$

$$\lim_{x\to\infty} \frac{e^x}{e^x - 2^x} = \lim_{x\to\infty} \frac{\frac{e^x}{e^x}}{\frac{e^x - 2^x}{e^x}} = \lim_{x\to\infty} \frac{1}{1-\left(\frac{2}{e}\right)^x} = \frac{1}{1-0} = \boxed{1}$$

$$\& \int_{1}^{\infty} \frac{1}{e^x} = \lim_{b\to\infty} \int_{1}^{b} \frac{1}{e^x}\,dx = \lim_{b\to\infty} \left[-e^{-x} \Big|_{1}^{b} \right] = \lim_{b\to\infty} \left[-e^{-b} + \frac{1}{e} \right]$$

$$= 0 + \frac{1}{e} = \boxed{\frac{1}{e}}$$

$$\Rightarrow \int_1^\infty \frac{1}{e^x}\,dx \text{ converges} \Rightarrow \int_1^\infty \frac{1}{e^x - 2^x}\,dx \text{ converges}$$

(by Limit Comparison Test).

65) Find the values of P for which each integral converges.

a) $\int_1^2 \frac{dx}{x(\ln x)^P}$, b) $\int_2^\infty \frac{dx}{x(\ln x)^P}$

let $u = \ln x$, when $x = 1 \Rightarrow u = 0$.

$du = \frac{dx}{x}$, $x = 2 \Rightarrow u = \ln 2$.

$x = \infty \Rightarrow u = \infty$

a) $\int_0^{\ln 2} \frac{du}{u^P}$, b) $\int_{\ln 2}^{\infty} \frac{du}{u^P}$

a) $\int_0^{\ln 2} \frac{du}{u^P}$

for $P = 1 \Rightarrow \int_0^{\ln 2} \frac{du}{u} = \lim_{a \to 0^+} \int_a^{\ln 2} \frac{du}{u} = \lim_{a \to 0^+} \ln|u| \Big|_a^{\ln 2}$

$= \lim_{a \to 0^+} (\ln 2 - \ln a)$

$\boxed{= \infty}$

for $P \neq 1 \Rightarrow \int_0^{\ln 2} \frac{du}{u^P} = \lim_{a \to 0^+} \int_a^{\ln 2} \frac{du}{u^P} = \lim_{a \to 0^+} \frac{u^{-P+1}}{-P+1} \Big|_a^{\ln 2}$

$= \lim_{a \to 0^+} \left(\frac{(\ln 2)^{1-P}}{1-P} - \frac{a^{1-P}}{1-P} \right)$

$= \begin{cases} \frac{(\ln 2)^{1-P}}{1-P} , & P < 1 \\ \infty , & P > 1 \end{cases}$

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$$\boxed{\text{So, } \int_1^2 \frac{dx}{x(\ln x)^p} = \begin{cases} \dfrac{(\ln 2)^{1-p}}{1-p} & , p<1 \\ \infty & , p \geq 1 \end{cases}}$$

Converges for $p<1$.

diverges for $p \geq 1$.

b) $\displaystyle\int_{\ln 2}^{\infty} \frac{du}{u^p}$

for $p=1 \Rightarrow \displaystyle\int_{\ln 2}^{\infty} \frac{du}{u} = \lim_{b\to\infty} \int_{\ln 2}^{b} \frac{du}{u} = \lim_{b\to\infty} \ln|u| \Big|_{\ln 2}^{b}$

$$= \lim_{b\to\infty} \ln b - \ln 2$$

$$= \infty .$$

for $p \neq 1 \Rightarrow \displaystyle\int_{\ln 2}^{\infty} \frac{du}{u^p} = \lim_{b\to\infty} \int_{\ln 2}^{b} \frac{du}{u^p}$

$$= \lim_{b\to\infty} \frac{u^{-p+1}}{-p+1} \Bigg|_{\ln 2}^{b}$$

$$= \lim_{b\to\infty} \frac{b^{-p+1}}{-p+1} - \frac{\ln 2^{-p+1}}{-p+1}$$

$$= \begin{cases} \dfrac{(\ln 2)^{1-p}}{p-1} & , p>1 \\ \infty & , p<1 \end{cases}$$

$$\boxed{\text{So, } \int_2^{\infty} \frac{dx}{x(\ln x)^p} = \begin{cases} \dfrac{(\ln 2)^{1-p}}{p-1} & , p>1 \\ \infty & , p\leq 1. \end{cases}}$$

converges for $p>1$.

diverges for $p \leq 1$.

10.1 Sequences.

6. $a_n = \frac{2^n - 1}{2^n}$. Find the values of a_1, a_2, a_3 & a_4.

$a_1 = \frac{2^1 - 1}{2^1} = \frac{1}{2}$, $a_2 = \frac{3}{4}$, $a_3 = \frac{7}{8}$, $a_4 = \frac{15}{16}$, ...

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10. $a_1 = -2$, $a_{n+1} = \frac{n a_n}{n+1}$, write the first ten terms of the sequence.

$a_1 = -2$, $\boxed{n=1}$ $a_{1+1} = \frac{1(a_1)}{1+1}$

$a_2 = \frac{-2}{2} = -1$

$\boxed{n=2}$ $a_{2+1} = \frac{2\, a_2}{2+1} = \frac{2(-1)}{3}$

$a_3 = -\frac{2}{3}$

$\boxed{n=3}$ $\frac{3\, a_3}{3+1} = a_{3+1}$

$a_4 = \frac{3(-\frac{2}{3})}{4} = -\frac{1}{2}$

$\boxed{n=4}$ $a_{4+1} = \frac{4\, a_4}{4+1} = \frac{4(-\frac{1}{2})}{5} = -\frac{2}{5}$

⋮

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22. 2, 6, 10, 14, 18, ... Find a formula for nth term of the sequence.

$a_1 = 2$

$\left\{\begin{array}{l} 6-2=4 \\ 10-6=4 \\ 14-10=4 \end{array}\right. \Rightarrow d=4.$

So, $a_n = 2 + (n-1)4$

$a_n = 2 + 4n - 4$

$a_n = -2 + 4n$. $n = 1, 2, \ldots$

Note arithmetic sequence

$a_n = a_1 + (n-1)d$

d: common difference

26 $0, 1, 1, 2, 2, 3, 3, \ldots$, find a formula for nth term of the sequence

$\left[\frac{1}{2}\right] = 0$

$\left[\frac{2}{2}\right] = [1] = 1$

$\left[\frac{3}{2}\right] = 1$

$\left[\frac{4}{2}\right] = [2] = 2$

$\left[\frac{5}{2}\right] = 2$

$\vdots$

So $a_n = \left[\frac{n}{2}\right]$. $\quad n = 1, 2, \ldots$

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31) $a_n = \dfrac{1 - 5n^4}{n^4 + 8n^3}$, Is the sequence converges or diverges, find its limit?

$$a_n = \frac{\frac{1}{n^4} - 5}{1 + \frac{8}{n}}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\frac{1}{n^4} - 5}{1 + \frac{8}{n}}$$

$$= \frac{-5}{1} = -5 \quad \text{Converges.}$$

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35) $a_n = 1 + (-1)^n$

$a_n = 0, 2, 0, 2, \ldots$

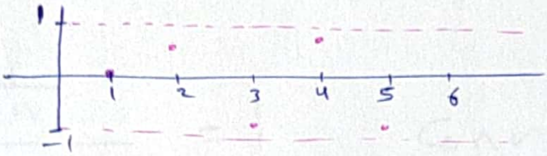
$\lim_{n\to\infty} a_n$ does not exist.

diverges.

36 $a_n = (-1)^n \left(1 - \frac{1}{n}\right)$.

$a_n = 0, \frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots$

$\lim_{n\to\infty} a_n$ does not exist. $\Rightarrow$ diverges



41 $a_n = \sqrt{\frac{2n}{n-1}}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{\frac{2n}{n-1}} = \sqrt{\lim_{n\to\infty} \frac{2n}{n-1}} = \sqrt{2}$. Converges.

44 $a_n = n\pi \cos(n\pi)$

$a_n = -\pi, 2\pi, -3\pi, 4\pi, \ldots$, $\lim_{n\to\infty} n\pi \cos(n\pi) = \lim_{n\to\infty} n\pi(-1)^n$

$\lim_{n\to\infty} a_n$ does not exist $\Rightarrow$ diverges.

48 $a_n = \frac{3^n}{n^3}$

$\lim_{n\to\infty} \frac{3^n}{n^3}$ $\frac{\infty}{\infty}$

$= \lim_{n\to\infty} \frac{\ln 3 \cdot 3^n}{3n^2}$

$= \lim_{n\to\infty} \frac{(\ln 3)^2\, 3^n}{6n}$

$= \lim_{n\to\infty} \frac{(\ln 3)^3\, 3^n}{6} = \infty$

diverges.

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50) $a_n = \dfrac{\ln n}{\ln 2n}$

$$\lim_{n\to\infty} \frac{\ln n}{\ln 2n} \quad \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{2}{2n}} = 1 \quad \text{Converges.}$$

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54) $a_n = \left(1 - \dfrac{1}{n}\right)^n$

$$a_n = \left(1 + \frac{(-1)}{n}\right)^n \qquad \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$$

$$\lim_{n\to\infty}\left(1+\frac{(-1)}{n}\right)^n = e^{-1} = \frac{1}{e} \quad \text{(Converges).}$$

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70) $\displaystyle\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n$

$$= \lim_{n\to\infty} e^{n\ln\left(\frac{n}{n+1}\right)} = \lim_{n\to\infty} e^{n\left(\ln n - \ln(n+1)\right)}$$

$$= \lim_{n\to\infty} e^{\left[\frac{\ln n - \ln(n+1)}{\frac{1}{n}}\right]}$$

$$= \lim_{n\to\infty} e^{\left[\frac{\frac{1}{n} - \frac{1}{n+1}}{-\frac{1}{n^2}}\right]} = \lim_{n\to\infty} e^{\left[-n^2\left(\frac{n+1-n}{n(n+1)}\right)\right]}$$

$$= \lim_{n\to\infty} e^{\frac{-n^2}{n(n+1)}}$$

$$= e^{-1} = \frac{1}{e} \quad \text{Converges.}$$

(58) $a_n = (n+4)^{\frac{1}{n+4}}$

let $u = n+4$.

$n \to \infty \Rightarrow u \to \infty$.

$\lim_{u \to \infty} u^{\frac{1}{u}} = 1$

So $\lim_{n \to \infty} a_n = \lim_{n \to \infty} (n+4)^{\frac{1}{n+4}} = 1$

Converges.

---

(60) $a_n = \ln n - \ln(n+1)$.

$a_n = \ln\left(\frac{n}{n+1}\right)$

$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \ln\left(\frac{n}{n+1}\right)$

$= \ln\left(\lim_{n \to \infty}\left(\frac{n}{n+1}\right)\right)$

$= \ln(1) = 0$. Converges.

---

(63) $a_n = \frac{n!}{n^n}$

$a_n = \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdot \cdots \frac{2}{n} \cdot \frac{1}{n}$.

$0 \leq a_n \leq \frac{1}{n}$.

$\lim_{n \to \infty} 0 \leq \lim_{n \to \infty} a_n \leq \lim_{n \to \infty} \frac{1}{n}$

$0 \qquad\qquad 0$

by Sandwich Theorem $\lim_{n \to \infty} a_n = 0$.

Converges.

72) $a_n = \left(1 - \frac{1}{n^2}\right)^n$

$$a_n = \left[\left(1 - \frac{1}{n}\right)\left(1 + \frac{1}{n}\right)\right]^n$$

$$= \left(1 + \frac{-1}{n}\right)^n \left(1 + \frac{1}{n}\right)^n$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1 + \frac{-1}{n}\right)^n \cdot \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

remember: $\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x$ x is fixed

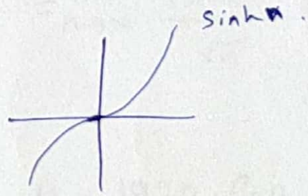
$$= e^{-1} \cdot e^{1}$$

$$= e^0$$

$$= \boxed{1}$$

---

76) $a_n = \sinh(\ln n)$



$$\sinh n = \frac{e^n - e^{-n}}{2}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sinh(\ln n)$$

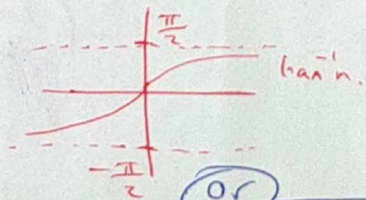
$$= \lim_{n\to\infty} \frac{e^{\ln n} - e^{-\ln n}}{2}$$

$$= \lim_{n\to\infty} \frac{n - \frac{1}{n}}{2}$$

$$= \infty \quad \text{diverges.}$$

---

82) $a_n = \frac{\tan^{-1} n}{\sqrt{n}}$



$$-\frac{\pi}{2} < \tan^{-1} n < \frac{\pi}{2}$$

$$-\frac{\pi}{2\sqrt{n}} < \frac{\tan^{-1} n}{\sqrt{n}} < \frac{\pi}{2\sqrt{n}}$$

$\lim_{n\to\infty} -\frac{\pi}{2\sqrt{n}} = 0$

& $\lim_{n\to\infty} \frac{\pi}{2\sqrt{n}} = 0$

So, by sandwich Theorem $\lim_{n\to\infty} \frac{\tan^{-1} n}{\sqrt{n}} = 0.$

or

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \cdot \tan^{-1} n$$

$$= 0 \cdot \frac{\pi}{2}$$

$$= 0.$$

Converges.

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86) $a_n = \frac{(\ln n)^5}{\sqrt{n}}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(\ln n)^5}{\sqrt{n}} \quad \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{\frac{5(\ln n)^4}{n}}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{10(\ln n)^4}{\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{\frac{40(\ln n)^3}{n}}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{80(\ln n)^3}{\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{\frac{240(\ln n)^2}{n}}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{480(\ln n)^2}{\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{\frac{960 \ln n}{n}}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{1920 \ln n}{\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{\frac{1920}{n}}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{3840}{\sqrt{n}}$$

$$= \boxed{0}$$ Converges.

---

92) $a_{n+1} = \frac{a_n+6}{a_n+2}$, $a_1 = -1$. (assume the sequence is converges & find the limit).

Since a_n converges $\Rightarrow \lim_{n\to\infty} a_n = L$

$$\& \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{a_n+6}{a_n+2}$$

$$L = \frac{L+6}{L+2}$$

$$L(L+2) = L+6$$

$$L^2 + 2L - L - 6 = 0$$

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$L^2 + 6 - 6 = 0.$

$(L+3)(L-2) = 0$

$L = -3$ or $L = 2.$

Since $a_n > 0$ for $n \geq 2$

So, $\boxed{L = 2}$

III) Determine if the following sequence is monotonic & if it is bounded.

$$a_n = \frac{3n+1}{n+1} \qquad \left[2, \frac{7}{3}, \frac{10}{4}, \frac{13}{5}, \dots\right]$$

$a_{n+1} > a_n.$

$$\frac{3(n+1)+1}{(n+1)+1} > \frac{3n+1}{n+1}$$

$$\frac{3n+4}{n+2} > \frac{3n+1}{n+1}$$

$3n^2+7n+4 > 3n^2+7n+2.$

$4 > 2$ ✓

$\therefore$ The sequence is nondecreasing (monotonic).

Now, $\frac{3n+1}{n+1} < 3.$

$3n+1 < 3n+3$

$1 < 3$

$\therefore$ The sequence is bounded above by 3 & bounded below by 2.

$\Rightarrow a_n$ is bounded.

10.2 Infinite Series.

[6] Find a formula for the nth partial sum & use it to find the series' sum:-

$$\frac{5}{1}+\frac{5}{2(3)}+\frac{5}{3(4)}+\dots+\frac{5}{n(n+1)}+\dots$$

$$a_n=\frac{5}{n(n+1)}=\frac{A}{n}+\frac{B}{n+1}$$

$\star$ $A=5$ & $B=-5$

So, $\frac{5}{n(n+1)}=\frac{5}{n}-\frac{5}{n+1}$

$$S_n=\sum_{n=1}^{\infty}\left(\frac{5}{n}-\frac{5}{n+1}\right)$$ Telescoping..

$$S_n=\left(\frac{5}{1}-\cancel{\frac{5}{2}}\right)+\left(\cancel{\frac{5}{2}}-\cancel{\frac{5}{3}}\right)+\left(\cancel{\frac{5}{3}}-\cancel{\frac{5}{4}}\right)+\dots+\cancel{\frac{5}{n}}-\frac{5}{n+1}$$

$$\boxed{S_n=5-\frac{5}{n+1}}$$ The nth partial sum.

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left(5-\frac{5}{n+1}\right)=5.$$

So, $\sum_{n=1}^{\infty}\frac{5}{n(n+1)}=5$ "That is Converges to 5"

$\boxed{14}$ $\sum_{n=0}^{\infty} \frac{2^{n+1}}{5^n}$

$$\sum_{n=0}^{\infty} \frac{2^{n+1}}{5^n} = \frac{2}{1} + \frac{4}{5} + \frac{8}{25} + \frac{16}{125} + \dots$$

its 'geometric series' with $*a = 2$ & $*r = \frac{2}{5} < 1$

So, $\sum_{n=0}^{\infty} \frac{2^{n+1}}{5^n}$ converges to $\frac{a}{1-r} = \frac{2}{1-\frac{2}{5}} = \frac{10}{3}$

$$\sum_{n=0}^{\infty} \frac{2^{n+1}}{5^n} = \frac{10}{3}$$

$\boxed{18}$ $\left(-\frac{2}{3}\right)^2 + \left(\frac{-2}{3}\right)^3 + \left(\frac{-2}{3}\right)^4 + \dots$

$$= \sum_{n=2}^{\infty} \left(-\frac{2}{3}\right)^n$$

its a geometric series with $\boxed{a} = \left(-\frac{2}{3}\right)^2 = \frac{4}{9}$

& $\boxed{r} = -\frac{2}{3}$.

$|r| < 1$ $\left(\left|-\frac{2}{3}\right| < 1\right)$, so the series converges

$$\sum_{n=2}^{\infty} \left(\frac{-2}{3}\right)^n = \frac{a}{1-r} = \frac{\frac{4}{9}}{1-\left(-\frac{2}{3}\right)} = \frac{\frac{4}{9}}{1+\frac{2}{3}} = \frac{\frac{4}{9}}{\frac{5}{3}}$$

$$= \frac{4}{9} \cdot \frac{3}{5} = \boxed{\frac{4}{15}}$$

[24] Express the following as the ratio of two integers.

$$1.\overline{414} = 1 + 0.414 + 0.000414 + \dots$$

$$= 1 + \underbrace{\frac{414}{1000} + \frac{414}{1000000} + \dots}$$

* Geometric series, with $a = \frac{414}{1000}$ & $r = \frac{1}{1000}$

Converges to $\frac{a}{1-r} = \frac{\frac{414}{1000}}{1 - \frac{1}{1000}} = \frac{414}{999}$.

→ So, $1.\overline{414} = 1 + \frac{414}{999} = \frac{1413}{999}$.

---

[32] $\sum_{n=0}^{\infty} \frac{e^n}{e^n + n}$

* → by nth term test:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^n}{e^n + n} = \lim_{n\to\infty} \frac{e^n}{e^n + 1}$$

$$= \lim_{n\to\infty} \frac{e^n}{e^n} = \lim_{n\to\infty} 1 = 1 \neq 0.$$

$\therefore \sum_{n=0}^{\infty} \frac{e^n}{e^n + n}$ diverges by nth term test.

$\boxed{38}$ $\sum_{n=1}^{\infty} (\tan n - \tan(n-1))$ A telescoping series.

$S_n = (\tan 1 - \tan 0) + (\cancel{\tan 2} - \cancel{\tan 1}) + \cdots + (\tan n - \cancel{\tan(n-1)})$

$= -\tan 0 + \tan n.$

$= \tan n.$

by nth Partial Sum:-

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \tan n = DNE.$

$\therefore \sum_{n=1}^{\infty} (\tan n - \tan(n-1))$ diverges by n^{th} partial Sum.

$\boxed{44}$ $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$

$$\frac{2n+1}{n^2(n+1)^2} = \frac{A}{n} + \frac{B}{n^2} + \frac{C}{n+1} + \frac{D}{(n+1)^2}$$

$A=0, \quad B=1, \quad C=0, \quad D=-1.$

So, $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right)$

$S_n = \left(1 - \cancel{\frac{1}{2^2}}\right) + \left(\cancel{\frac{1}{2^2}} - \cancel{\frac{1}{3^2}}\right) + \cdots + \left(\cancel{\frac{1}{n^2}} - \frac{1}{(n+1)^2}\right)$

$S_n = 1 - \frac{1}{(n+1)^2}$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{(n+1)^2}\right)$$

$$= 1 - 0 = 1$$

So, the series converges to $\underline{1}$.

---

54 $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{5^n}$

$$= \frac{\cos(0)}{1} + \frac{\cos(\pi)}{5} + \frac{\cos(2\pi)}{5^2} + \ldots$$

$$= 1 - \frac{1}{5} + \frac{1}{5^2} + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{5^n} = \sum_{n=0}^{\infty}\left(\frac{-1}{5}\right)^n.$$

Geometric series with $a = 1$ & $r = -\frac{1}{5}$.

$|r| = \left|-\frac{1}{5}\right| < 1$, so its converges to $\frac{a}{1-r}$

$$\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{5^n} = \frac{1}{1 - -\frac{1}{5}} = \frac{1}{\frac{6}{5}} = \boxed{\frac{5}{6}}$$

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$\boxed{62}$ $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

by nth term test

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n^n}{n!} = \lim_{n \to \infty} \frac{n \cdot n \cdot \ldots \cdot n}{n \cdot (n-1), \ldots \cdot 1}$$

$$= \lim_{n \to \infty} \frac{n}{n} \cdot \lim_{n \to \infty} \frac{n}{n-1} \cdot \ldots \cdot \lim_{n \to \infty} \frac{n}{1}$$

$$= (1) \quad (1) \ldots \infty$$

$$= \infty \neq 0.$$

$\therefore \sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges by nth term test.

$\boxed{63}$ $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n}$

$$= \sum_{n=1}^{\infty} \left(\frac{2}{4}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n.$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n.$$

$$= \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right) + \left(\frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \ldots\right).$$

geometric serieses

$a = \frac{1}{2}$ $\qquad$ $a = \frac{3}{4}$.

$r = \frac{1}{2} < 1$ $\qquad$ $r = \frac{3}{4} < 1$

$$= \frac{\frac{1}{2}}{1-\frac{1}{2}} + \frac{\frac{3}{4}}{1-\frac{3}{4}}$$

$$= 1 + 3 = \boxed{4}$$

$\boxed{78}$ Find the values of x for which the series converges & find the sum.

$$\sum_{n=0}^{\infty} (\ln x)^n$$

$$= 1 + \ln x + (\ln x)^2 + \dots$$

geometric series with $\boxed{a = 1}$ & $\boxed{r = \ln x}$

→ To be converges:

$$|r| < 1$$

$$\rightarrow -1 < r < 1$$

$$-1 < \ln x < 1$$

$$e^{-1} < x < e^{1}$$

So, if $\frac{1}{e} < x < e$, then the series $\sum_{n=0}^{\infty} (\ln x)^n$ converges to $\frac{a}{1-r} = \frac{1}{1-\ln x}$

90 Find the values of b for which

$$1 + e^{b} + e^{2b} + e^{3b} + \cdots = 9.$$

$$\sum_{n=0}^{\infty} (e^{b})^{n} = 9.$$

geometric series with $a = 1$ & $r = e^{b}$.

$$\sum_{n=0}^{\infty} (e^{b})^{n} = \frac{1}{1 - e^{b}} = 9.$$

$$1 = 9 - 9e^{b}.$$

$$9e^{b} = 8.$$

$$e^{b} = \frac{8}{9}$$

$$b = \ln\left(\frac{8}{9}\right).$$

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10.3 The Integral Test.

6 $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$, use the integral test.

let $f(x) = \frac{1}{x(\ln x)^2}$, $f(x)$ is +ve , continuous & decreasing $\forall x \geq 2$

★ $f(x)$ is decreasing since:

$$f'(x) = \frac{-\left[2\frac{x \ln x}{x} + (\ln x)^2\right]}{x^2(\ln x)^4} = \frac{-2\ln x}{x^2(\ln x)^4} - \frac{(\ln x)^2}{x^2(\ln x)^4}$$

$$= \frac{-2}{x^2(\ln x)^3} - \frac{1}{x^2(\ln x)^2}$$

f' sign 2 - - - - →

Now, To find $\int_2^{\infty} \frac{1}{x(\ln x)^2} dx$

let $u = \ln x \Rightarrow du = \frac{dx}{x}$

when $x = 2 \Rightarrow u = \ln 2$.
$x = \infty \Rightarrow u = \infty$

$$\Rightarrow \int_{\ln 2}^{\infty} \frac{1}{u^2} du = \lim_{b \to \infty} \int_{\ln 2}^{b} u^{-2} du$$

$$= \lim_{b \to \infty} \left(-\frac{1}{u} \Big|_{\ln 2}^{b} \right) = \lim_{b \to \infty} \left(-\frac{1}{b} + \frac{1}{\ln 2} \right)$$

$$= \frac{1}{\ln 2}$$ Converges.

$\therefore \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges by integral test.

---

13 $\sum_{n=1}^{\infty} \frac{n}{n+1}$

$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n}{n+1} = 1 \neq 0$.

by n-th term test $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges.

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$\boxed{20}$ $\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$

*remember :-
$\ln x > 0 \quad \forall x > 1$.

Try to use the integral test.

let $f(x) = \frac{\ln x}{\sqrt{x}}$, $f(x)$ is positive, continuous & decreasing $\forall x > 8$.

Since $f'(x) = \frac{\frac{\sqrt{x}}{x} - \frac{\ln x}{2\sqrt{x}}}{(\sqrt{x})^2} = \frac{2 - \ln x}{2x^{3/2}}$

$2 - \ln x < 0$
$\ln x > 2$.
$x > e^2$

f' sign: $+$ on $(2, e^2)$, 0 at e^2, $-$ after e^2

$e^2 \simeq 7.4$.

$$\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}} = \sum_{n=2}^{7} \frac{\ln n}{\sqrt{n}} + \sum_{n=8}^{\infty} \frac{\ln n}{\sqrt{n}}.$$

So, we can apply the integral test for $n \geq 8$.

Now, $\int_8^{\infty} \frac{\ln x}{\sqrt{x}} dx$.

let $u = \ln x \Rightarrow du = \frac{dx}{x}$

$e^u = x \Rightarrow e^u du = dx$.

$e^{u/2} = \sqrt{x}$

when $x = 8 \Rightarrow u = \ln 8$.

$x = \infty \Rightarrow u = \infty$.

$$= \int_{\ln 8}^{\infty} \frac{u}{e^{u/2}} \cdot e^u du. = \int_{\ln 8}^{\infty} u e^{u/2} du.$$

$$= \lim_{b \to \infty} \int_{\ln 8}^{b} u e^{u/2} du.$$

deriv	Integral
u	$e^{u/2}$
1	$+$ $2e^{u/2}$
0	$-$ $4e^{u/2}$

$$= \lim_{b \to \infty} \left(\left(2ue^{u/2} - 4e^{u/2} \right) \Big|_{\ln 8}^{b} \right)$$

$$= \lim_{b \to \infty} \left(2e^{u/2}(u-2) \Big|_{\ln 8}^{b} \right).$$

$$= \lim_{b \to \infty} \left(2e^{b/2}(b-2) - 2e^{\ln 8/2}(\ln 8 - 2) \right).$$

$= \infty$ diverges.

by the integral test $\sum_{n=8}^{\infty} \frac{\ln}{\sqrt{n}}$ diverges.

So, $\sum_{n=2}^{\infty} \frac{\ln 2}{\sqrt{n}}$ diverges.

$\boxed{22}$ $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{4^n+3}$

Use n-th term test:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5^n}{4^n+3} \quad \frac{\infty}{\infty}.$$

$$= \lim_{n\to\infty} \frac{\ln 5 \; 5^n}{\ln 4 \; 4^n + 0} = \frac{\ln 5}{\ln 4} \lim_{n\to\infty} \left(\frac{5}{4}\right)^n \qquad \frac{5}{4} > 1.$$

$$= \infty \neq 0.$$

$\Rightarrow$ by n-th term test the series diverges.

$\boxed{28}$ $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$

by using n-th term test:

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e^1 = e \neq 0.$$

So the series diverges.

$\boxed{32}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(1+\ln^2 n)}$

let $f(x) = \dfrac{1}{x(1+\ln^2 x)}$ [positive, continuous & decreasing for $x \geq 1$]

$$f'(x) = -\frac{\left(x \cdot 2\ln x \cdot \frac{1}{x} + (1+\ln^2 x)\right)}{x^2(1+\ln^2 x)^2} = \frac{-2\ln x - (1+\ln^2 x)}{x^2(1+\ln^2 x)^2}.$$

$$= -\frac{2\ln x}{x^2(1+\ln^2 x)^2} - \frac{1}{x^2(1+\ln^2 x)}$$

let $u = \ln x \Rightarrow du = \dfrac{dx}{x}$

when $x=1 \Rightarrow u=0$.

$x = \infty \Rightarrow u = \infty$.

Now, $\displaystyle\int_1^{\infty} \frac{1}{x(1+\ln^2 x)}\,dx$

$$= \int_0^{\infty} \frac{du}{1+u^2} = \lim_{b\to\infty} \int_0^b \frac{du}{1+u^2}$$

$$= \lim_{b\to\infty} \left(\tan^{-1} u \Big|_0^b\right) = \lim_{b\to\infty} \left(\tan^{-1} b - \tan^{-1} 0\right) = \frac{\pi}{2} - 0 = \frac{\pi}{2} \quad \text{Converges.}$$

So, by the integral test $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(1+\ln^2 n)}$ Converges.

$\boxed{38}$ $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$.

let $f(x) = \frac{x}{x^2+1}$

$f'(x) = \frac{(x^2+1) - 2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$ f' sign $\vert$ — — — — $\rightarrow$

So, $f(x)$ is continuous, positive & decreasing $\forall x \geq 1$.

$\int_1^{\infty} \frac{x}{x^2+1} = \lim_{b\to\infty} \int_1^b \frac{x}{x^2+1}$

$= \lim_{b\to\infty} \left(\frac{1}{2} \ln |x^2+1| \Big|_1^b \right) = \lim_{b\to\infty} \left(\frac{1}{2} \ln (b^2+1) - \frac{1}{2} \ln 2 \right)$

$= \infty$ diverges.

So, by integral test the series diverges

$\boxed{40}$ $\sum_{n=1}^{\infty} \operatorname{sech}^2 n$.

let $f(x) = \operatorname{sech}^2 x = \frac{1}{\cosh^2 x} = \left(\frac{2}{e^x + e^{-x}} \right)^2 = \frac{4}{(e^x+e^{-x})^2}$

$f'(x) = -\frac{8(e^x - e^{-x})}{(e^x+e^{-x})^3}$

So, $f(x)$ is positive, continuous & decreasing $\forall x \geq 1$.

Now, $\int_1^{\infty} \operatorname{sech}^2 x \, dx = \lim_{b\to\infty} \int_1^b \operatorname{sech}^2 x \, dx$.

$= \lim_{b\to\infty} \left(\tanh x \Big|_1^b \right) = \lim_{b\to\infty} (\tanh b - \tanh 1)$

$= 1 - \tanh 1$

converges.

So, by the integral test

$\sum_{n=1}^{\infty} \operatorname{sech}^2 n$ converges.

$\star$ remember:-

$\tanh b = \frac{e^b - e^{-b}}{e^b + e^{-b}}$

$\lim_{b\to\infty} \frac{1 - \frac{e^{-b}}{e^b}}{1 + \frac{e^{-b}}{e^b}} = \frac{1-0}{1+0} = \boxed{1}$

(42) For what values of a do the series converge??

$$\sum_{n=3}^{\infty}\left(\frac{1}{n-1}-\frac{2a}{n+1}\right)$$

by using the integral test:-

$$\int_3^{\infty}\left(\frac{1}{x-1}-\frac{2a}{x+1}\right)dx = \lim_{b\to\infty}\int_3^b\left(\frac{1}{x-1}-\frac{2a}{x+1}\right)dx.$$

$$= \lim_{b\to\infty}\left(\ln|x-1|-2a\ln|x+1|\Big|_3^b\right)$$

$$= \lim_{b\to\infty}\left(\ln\left|\frac{x-1}{(x+1)^{2a}}\right|\Big|_3^b\right)$$

$$= \lim_{b\to\infty}\left(\ln\left|\frac{b-1}{(b+1)^{2a}}\right|-\ln\left|\frac{2}{4^{2a}}\right|\right)$$

$$\lim_{b\to\infty}\ln\left(\frac{b-1}{(b+1)^{2a}}\right)=\lim_{b\to\infty}\ln\frac{1}{2a(b+1)^{2a-1}}$$

$$=\begin{cases}0 & , a=\frac{1}{2}\\ \infty & , a<\frac{1}{2}\end{cases}$$

If $a>\frac{1}{2}$, the terms of the series become negative & the integral test does not apply.

* but when $a>\frac{1}{2}$, the series behaves like a negative multiple of the harmonic series, & so, it's diverges.

So, $\sum_{n=3}^{\infty}\frac{1}{n-1}-\frac{2a}{n+1}$ converges only when $a=\frac{1}{2}$

10.4 :- Comparsion Tests

8 Use D.C.T to determine if the following sequence converges or diverges.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{\sqrt{n^2+3}}$$

Compare with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Note that :- $\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{\sqrt{n^2+3}}$ & $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ have nonnegative terms.

$\Rightarrow$ for $n \geq 1 \Rightarrow \sqrt{n} \geq 1$

$$2\sqrt{n} \geq 2$$

$$(2\sqrt{n}+1 \geq 3) \times n.$$

$$2n\sqrt{n}+n \geq 3n \geq 3 \qquad \text{add } n^2 \text{ for both sides.}$$

$$2n\sqrt{n}+n+n^2 \geq n^2+3$$

$$n(n+2\sqrt{n}+1) \geq n^2+3.$$

$$n(\sqrt{n}+1)^2 \geq n^2+3$$

$$\frac{n(\sqrt{n}+1)^2}{n^2+3} \geq 1 \qquad \text{take the square root for the both sides.}$$

$$\frac{\sqrt{n}(\sqrt{n}+1)}{\sqrt{n^2+3}} \geq 1.$$

$$\frac{\sqrt{n}+1}{\sqrt{n^2+3}} \geq \frac{1}{\sqrt{n}}$$

(both series have nonnegative terms for $n \geq 1$).

but, we know that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series, since $p = \frac{1}{2} \leq 1$

So, by D.C.T $\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{\sqrt{n^2+3}}$ diverges.

(15) Use L.C.T to determine if the following series converges or diverges.
$$\sum_{n=2}^{\infty} \frac{1}{\ln n}$$

* Compare with $\sum_{n=2}^{\infty} \frac{1}{n}$, which is divergent p-series, since $p = 1 \leq 1$.

* $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ & $\sum_{n=2}^{\infty} \frac{1}{n}$ have positive terms for $n \geq 2$.

* Now, $\lim_{n\to\infty} \frac{\frac{1}{\ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\ln n} = \lim_{n\to\infty} \frac{1}{1/n} = \infty$.

$\Rightarrow$ by L.C.T $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges.

---

(18) Use any method to determine if the following series converges or diverges
$$\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$$

* Let use L.C.T & compare with $\sum_{n=1}^{\infty} \frac{1}{n}$

* Both series have positive terms.

* $\sum_{n=1}^{\infty} \frac{1}{n}$ divergent (harmonic series).

* $\lim_{n\to\infty} \frac{\frac{3}{n+\sqrt{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{3n}{n+\sqrt{n}} = 3 > 0$.

So, both series converge or both diverge.

but, we say that, $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent

So, $\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$ is also divergent.
by L.C.T

By D.C.T

$n+n+n > n+\sqrt{n}+0$.

$3n > n+\sqrt{n}$.

$\frac{3}{n+\sqrt{n}} > \frac{1}{n}$

$\sum \frac{1}{n}$ dive

So by D.C.T $\sum \frac{3}{n+\sqrt{n}}$ div.

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(27) $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$

Compare with $\sum_{n=3}^{\infty} \frac{1}{n}$

Note that: $\sum_{n=3}^{\infty} \frac{1}{n}$ & $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$ have positive terms.

$\Rightarrow n > \ln n$ take ln for both sides.

$\ln n > \ln(\ln n)$

& $n > \ln n > \ln(\ln n)$.

So, $\frac{1}{n} < \frac{1}{\ln(\ln n)}$

$\sum_{n=3}^{\infty} \frac{1}{n} < \sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$

but, we know that $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges (harmonic series).

So, by D.C.T $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$ diverges.

---

(28) $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$.

Let use L.C.T & compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Note: $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$ & $\sum_{n=1}^{\infty} \frac{1}{n^2}$ have nonnegative terms

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (P-series, $p > 1$).

Now, $\lim_{n\to\infty} \frac{\frac{(\ln n)^2}{n^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{(\ln n)^2}{n^3} \cdot n^2 = \lim_{n\to\infty} \frac{(\ln n)^2}{n}$ $\quad \frac{\infty}{\infty}$

$$= \lim_{n\to\infty} \frac{2\ln n \cdot \left(\frac{1}{n}\right)}{1} = \lim_{n\to\infty} \frac{2\ln n}{n} \quad \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{2\left(\frac{1}{n}\right)}{1} = \lim_{n\to\infty} \frac{2}{n} = 0.$$

So, by L.C.T $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$ converges.

$\boxed{32}$ $\sum_{n=2}^{\infty} \frac{\ln(n+1)}{n+1}$

Use L.C.T & compare with $\sum_{n=2}^{\infty} \frac{1}{n+1}$

$$\lim_{n\to\infty} \frac{\frac{\ln(n+1)}{n+1}}{\frac{1}{n+1}} = \lim_{n\to\infty} \ln(n+1) = \infty.$$

$\sum_{n=2}^{\infty} \frac{1}{n+1}$ (use the integral test). $\Rightarrow f(x) = \frac{1}{x+1}$ (+ve, cont. & decreasing)

↳ have positive terms.

Now, $\int_2^{\infty} \frac{1}{x+1}\,dx = \lim_{b\to\infty} \int_2^b \frac{dx}{x+1} = \lim_{b\to\infty} \left(\ln|x+1| \Big|_2^b\right)$

$$= \lim_{b\to\infty} \left(\ln|b+1| - \ln 3\right)$$

$$= \infty \quad \text{diverges.}$$

So, $\sum_{n=2}^{\infty} \frac{1}{n+1}$ diverges by integral test.

So, by L.C.T $\sum_{n=2}^{\infty} \frac{\ln(n+1)}{n+1}$ diverges

---

Note: You can use the integral test to show that $\sum_{n=2}^{\infty} \frac{\ln(n+1)}{n+1}$ diverges.

explain: $\int_2^{\infty} \frac{\ln(x+1)}{x+1}\,dx = \int_{\ln 3}^{\infty} u\,du$

$$= \lim_{b\to\infty} \left[\frac{1}{2}u^2\right]_{\ln 3}^{b} = \lim_{b\to\infty} \frac{1}{2}\left(b^2 - \ln^2 3\right) = \infty$$

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$$\sum_{n=1}^{\infty} \frac{2^n + 3^n}{3^n + 4^n}$$

Let Compare with $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n}$ & use D.C.T.

* $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{3^n + 4^n}$ & $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n}$ have nonnegative terms.

Now, $\frac{2^n + 3^n}{3^n + 4^n} < \frac{2^n + 3^n}{4^n}$

$$\sum_{n=1}^{\infty} \frac{2^n + 3^n}{3^n + 4^n} < \sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n}.$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n} = \sum_{n=1}^{\infty} \left[\left(\frac{2}{4}\right)^n + \left(\frac{3}{4}\right)^n\right]$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$$ (Geometric serieses with $|r|<1$)

$$= \frac{\frac{1}{2}}{1-\frac{1}{2}} + \frac{\frac{3}{4}}{1-\frac{3}{4}}$$

$$= \frac{\frac{1}{2}}{\frac{1}{2}} + \frac{\frac{3}{4}}{\frac{1}{4}} = 1 + 3 = \boxed{4}$$

So, $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n}$ Converges.

By D.C.T $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{3^n + 4^n}$ converges.

Note :- We can use L.C.T

$$\lim_{n\to\infty} \frac{\frac{2^n + 3^n}{3^n + 4^n}}{\frac{2^n + 3^n}{4^n}} = \lim_{n\to\infty} \frac{4^n}{3^n + 4^n} = 1 > 0$$ (both converges)

43 $\sum_{n=2}^{\infty} \frac{1}{n!}$

Compare with

$\sum_{n=2}^{\infty} \frac{1}{n^2-n}$ & use D.C.T.

$\sum_{n=2}^{\infty} \frac{1}{n!} \leq \sum_{n=2}^{\infty} \frac{1}{n^2-n}$

$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n^2-n}$ converges.

So, $\sum_{n=2}^{\infty} \frac{1}{n!}$ converges.

$n! = n(n-1)(n-2)\dots 2\cdot 1$

$n! \geq n(n-1)$

$\frac{1}{n!} \leq \frac{1}{n(n-1)} = \frac{1}{n^2-n}$

$\sum_{n=2}^{\infty} \frac{1}{n!} \leq \sum_{n=2}^{\infty} \frac{1}{n^2-n}$

Now: Determine $\sum_{n=2}^{\infty} \frac{1}{n^2-n}$ converges or diverges.

$\rightarrow$ Compare with $\sum_{n=2}^{\infty} \frac{1}{n^2}$.

($\sum_{n=2}^{\infty} \frac{1}{n^2-n}$ & $\sum_{n=2}^{\infty} \frac{1}{n^2}$ have nonnegative terms.

& $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges (p-series, $p>1$)

by L.C.T

$$\lim_{n\to\infty} \frac{\frac{1}{n^2-n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-n} = 1 > 0$$

So, both converges.

$\sum_{n=2}^{\infty} \frac{1}{n^2-n}$ converges.

Note: You can use the ratio test.

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52) $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$

Compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, ($\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, p-series with $p>1$).

$\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$ & $\sum_{n=1}^{\infty} \frac{1}{n^2}$ have nonnegative terms.

$$\lim_{n\to\infty} \frac{\frac{\sqrt[n]{n}}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{n^2} \cdot n^2$$

$$= \lim_{n\to\infty} \sqrt[n]{n} = 1 > 0$$

So, by L.C.T $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$ converges.

10.5 The ratio & Root tests.

$\boxed{7}$ Use the ratio test to determine if the following series converges or diverges

$$\sum_{n=1}^{\infty} \frac{n^2(n+2)!}{n!\,3^{2n}}$$

$\dfrac{n^2(n+2)!}{n!\,3^{2n}} > 0$ for all $n \geq 1$.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left[\frac{\dfrac{(n+1)^2((n+1)+2)!}{(n+1)!\,3^{2(n+1)}}}{\dfrac{n^2(n+2)!}{n!\,3^{2n}}} \right]$$

$$= \lim_{n\to\infty} \frac{(n+1)^2(n+3)!}{(n+1)!\,3^{2n+2}} \cdot \frac{n!\,3^{2n}}{n^2(n+2)!}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2(n+3)\cdot\cancel{(n+2)!}\cdot\cancel{n!}\,\cancel{3^{2n}}}{(n+1)\cdot\cancel{n!}\,\cancel{3^{2n}}\cdot 3^2\cdot n^2\,\cancel{(n+2)!}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2(n+3)}{9(n+1)\,n^2}$$

$$= \boxed{\frac{1}{9}} < 1$$

by ration test, the series is convergent.

(12) $\sum_{n=1}^{\infty} \left(\ln\left(e^2+\frac{1}{n}\right)\right)^{n+1}$ use root test.

$\left[\ln\left(e^2+\frac{1}{n}\right)\right]^{n+1} \geq 0$ for all $n \geq 1$

Now, $\lim_{n\to\infty} \sqrt[n]{\left[\ln\left(e^2+\frac{1}{n}\right)\right]^{n+1}}$

$= \lim_{n\to\infty} \left[\ln\left(e^2+\frac{1}{n}\right)\right]^{\frac{n+1}{n}}$

$= \lim_{n\to\infty} \left[\ln\left(e^2+\frac{1}{n}\right)\right]^{1+\frac{1}{n}}$

$= \left[\ln\left(e^2+0\right)\right]^{1+0}$

$= 2 > 1$

by The root test the series diverges.

---

(16) $\sum_{n=2}^{\infty} \frac{1}{n^{1+n}}$, use root test.

$\frac{1}{n^{1+n}} \geq 0$ for all $n \geq 2$.

$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n^{1+n}}} = \lim_{n\to\infty} \frac{\sqrt[n]{1}}{\sqrt[n]{n^{1+n}}}$

$= \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}} \cdot n^{\frac{n}{n}}}$

$= \lim_{n\to\infty} \frac{1}{\sqrt[n]{n} \cdot n} = 0 < 1$

So, by The root test, the series converges.

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20 Use any method to determine if the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n!}{10^n}$$

let use the ratio test.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{10^{n+1}}}{\frac{n!}{10^n}}$$

$$= \lim_{n\to\infty} \frac{(n+1).\cancel{n!}}{\cancel{10^n}.10} \cdot \frac{\cancel{10^n}}{\cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{(n+1)}{10} = \infty > 1$$

The series diverges by the ratio test.

---

28 $$\sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$$

by root test, $\lim_{n\to\infty} \sqrt[n]{\frac{(\ln n)^n}{n^n}} = \lim_{n\to\infty} \frac{\ln n}{n}$ $\frac{\infty}{\infty}$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0 < 1$$

So, The series converges.

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38) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Let use ratio test :>

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left[\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}\right]$$

$$= \lim_{n\to\infty} \frac{(n+1)\cancel{n!}}{(n+1)^n \cancel{(n+1)}} \cdot \frac{n^n}{\cancel{n!}} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n}$$

$$= \lim_{n\to\infty} \frac{1}{\frac{(n+1)^n}{n^n}} = \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e^1}$$

$$= \frac{1}{e} < 1$$

The series converges.

---

43) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

using ratio test :>

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{((n+1)!)^2}{(2(n+1))!} \cdot \frac{(2n)!}{(n!)^2}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2 \cancel{(n!)^2}}{(2n+2)(2n+1)\cancel{(2n)!}} \cdot \frac{\cancel{(2n)!}}{\cancel{(n!)^2}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1$$

So, the series converges.

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(46) $a_1 = 1$, $a_{n+1} = \frac{1 + \tan^{-1} n}{n} a_n$.

Let use ratio test.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(1+\tan^{-1} n)(a_n)}{n}}{a_n}$$

$$= \lim_{n\to\infty} \frac{(1+\tan^{-1} n)\cancel{a_n}}{n} \cdot \frac{1}{\cancel{a_n}}$$

$$= \frac{1+\frac{\pi}{2}}{\lim_{n\to\infty} n} = 0 < 1$$

∴ The series converges.

---

(60) $\sum_{n=1}^{\infty} \frac{n^n}{(2^n)^2}$

Using root test.

$$\lim_{n\to\infty} \sqrt[n]{\frac{n^n}{(2^n)^2}} = \lim_{n\to\infty} \frac{n^{\frac{n}{n}}}{2^{\frac{2n}{n}}}$$

$$= \lim_{n\to\infty} \frac{n}{2^2}$$

$$= \lim_{n\to\infty} \frac{n}{4}$$

$$= \infty$$

The series diverges.

10.6 Alternating Series, Absolute & Conditional Converges.

8 Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} (-1)^n \cdot \frac{10^n}{(n+1)!}$$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{10^n}{(n+1)!}$$

Using ratio test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{10^{n+1}}{(n+1+1)!} \cdot \frac{(n+1)!}{10^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{10^n} \cdot 10}{(n+2)\cancel{(n+1)!}} \cdot \frac{\cancel{(n+1)!}}{\cancel{10^n}}$$

$$= \lim_{n\to\infty} \frac{10}{n+2} = 0 < 1$$

So, $\sum_{n=1}^{\infty} |a_n|$ converges

The series converges absolutely $\Rightarrow \sum_{n=1}^{\infty} (-1)^n \frac{10^n}{(n+1)!}$ converges by the Absolute convergence test.

20 $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n!}{2^n}$$

by nth term test

$$\lim_{n\to\infty} \frac{n!}{2^n} = \frac{1}{\lim_{n\to\infty}\left(\frac{2^n}{n!}\right)} = \frac{1}{0} = \infty$$

The series diverges.

$\boxed{13}$ $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}+1}{n+1}$

Using the alternating series test :-

1] $u_n = \frac{\sqrt{n}+1}{n+1} > 0 \quad , \forall n \geq 1.$

2] $u_n \geq u_{n+1}$

let $f(x) = \frac{\sqrt{x}+1}{x+1}$, $x \geq 1.$

$$f'(x) = \frac{\frac{x+1}{2\sqrt{x}} - (\sqrt{x}+1)}{(x+1)^2}$$

$$= \frac{x+1-2x-2\sqrt{x}}{2\sqrt{x}\,(x+1)^2}$$

$$= \frac{-x-2\sqrt{x}+1}{2\sqrt{x}\,(x+1)^2} < 0 \quad , \forall x \geq 1$$

$f(x)$ is decreasing

3] $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{\sqrt{n}+1}{n+1} = \boxed{0}$

So, the series converges by alternating series test.

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25. Determine if the series converges absolutely, conditionally or diverges.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1+n}{n^2}$$

★ Converges Absolutely ??

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1+n}{n^2}$$

$$= \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\substack{\text{converges} \\ (\text{P-series } P>1)}} + \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\substack{\text{diverges} \\ (\text{harmonic series})}} = \text{diverges.}$$

$\Rightarrow$ The series does not converge absolutely.

★ Converges Conditionally ??

$\boxed{1}$ let $U_n = \dfrac{1+n}{n^2} > 0$

$f(x) = \dfrac{1+x}{x^2} = \dfrac{1}{x^2} + \dfrac{1}{x}$, $x \geq 1$.

$\boxed{2} \Rightarrow U_n > U_{n+1} > 0 \quad \forall n \geq 1.$

$f'(x) = -\left(\dfrac{2}{x^3} + \dfrac{1}{x^2}\right) < 0 \quad \forall x \geq 1.$

$f(x)$ is decreasing.

$\boxed{3}$ & $\lim_{n\to\infty} U_n = \lim_{n\to\infty} \dfrac{1+n}{n^2} = 0$

$\Rightarrow$ Converges by alternating series test.

$\Rightarrow$ So, The series converges conditionally but not converges absolutely.

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(30) $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n - \ln n}$

* Converges absolutely?? (No)

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{\ln n}{n - \ln n}$$

let use comparison test & compare with $\frac{1}{n}$.

$n - \ln n < n$.

$$\frac{1}{n - \ln n} > \frac{1}{n} \Rightarrow \frac{\ln n}{n - \ln n} > \frac{1}{n}$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ (diverges - harmonic series).

So, by direct comparison test $\sum_{n=1}^{\infty} \frac{\ln n}{n - \ln n}$ diverges.

* Converges conditionally?? (Yes).

1] $u_n = \frac{\ln n}{n - \ln n} \geq 0 \quad \forall n \geq 1.$

2] $f(x) = \frac{\ln x}{x - \ln x}$, $f'(x) = \frac{\frac{x - \ln x}{x} - \ln x \left(1 - \frac{1}{x}\right)}{(x - \ln x)^2}$.

$$= \frac{x - \cancel{\ln x} - x\ln x - \cancel{\ln x}}{x\,(x - \ln x)^2}$$

$$= \frac{\cancel{x}(1 - \ln x)}{\cancel{x}\,(x - \ln x)^2} < 0 \quad \forall x > e$$

So, $u_n \geq u_{n+1} \quad \forall n \geq 3$.

3] Now, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{\ln n}{n - \ln n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1 - \frac{1}{n}} = 0$

$\Rightarrow \sum (-1)^n \frac{\ln n}{n - \ln n}$ (converges by alternating series test).

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39 $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{2^n\, n!\, n}$

diverges by n-th term test

since $\displaystyle\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{(2n)!}{2^n\, n!\, n}$

$$= \lim_{n\to\infty} \frac{(2n)(2n-1)(2n-2)\cdots(n+1)\cancel{n!}}{2^n\, \cancel{n!}\, \cancel{n}}$$

$$= \lim_{n\to\infty} \frac{(2n-1)(2n-2)\cdots(n+1)}{2^{n-1}}$$

$$> \lim_{n\to\infty} \left(\frac{n+1}{2}\right)^{n-1}$$

$$= \infty \neq 0.$$

---

42 $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n^2+n} - n\right)$

Let use n-th term test

$$\lim_{n\to\infty} \left(\sqrt{n^2+n} - n\right) = \lim_{n\to\infty} \left(\sqrt{n^2+n} - n\right) \cdot \frac{\sqrt{n^2+n} + n}{\sqrt{n^2+n} + n}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2} + n - \cancel{n^2}}{\sqrt{n^2+n} + n} = \lim_{n\to\infty} \frac{n}{|n|\sqrt{1+\frac{1}{n}} + n}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}\left(\sqrt{1+\frac{1}{n}} + 1\right)} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}} + 1}$$

$$= \frac{1}{1+1} = \frac{1}{2} \neq 0$$

The series diverges.

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50 Estimate the magnitude of the error involved in using the sum of the first four terms to approximate the sum of the series.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{10^n}$$

$$|\text{error}| < u_{4+1} = |a_{4+1}|$$

$$|\text{error}| < |a_5| = \left| (-1)^{5+1} \frac{1}{10^5} \right|$$

$$|\text{error}| < 0.00001$$

54 Determine how many terms should be used to estimate the sum of the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$ with an error of less than 0.001

$$|\text{error}| < 0.001$$

$$\Rightarrow u_{n+1} < 0.001$$

$$\frac{n+1}{(n+1)^2+1} < 0.001$$

$$(n+1)\, 1000 < (n+1)^2 + 1$$

$$1000n + 1000 < n^2 + 2n + 2$$

$$0 < n^2 - 998n - 998$$

$$n > \frac{-(-998) \mp \sqrt{(-998)^2 - 4(1)(-998)}}{2(1)} \approx 998.99899$$

$$n \geq 999$$

10.7 : Power Series

a Find the series' radius & interval of convergence.

b For what values of x does the series converges absolutely?

c For what values of x does the series converges conditionally?

4 $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$

Ratio test :⇒ The series converges absolutely if ⇒

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$$

$$\lim_{n\to\infty} \left|\frac{\frac{(3x-2)^{n+1}}{n+1}}{\frac{(3x-2)^n}{n}}\right| = \lim_{n\to\infty} \left|\frac{\cancel{(3x-2)^n}\,(3x-2)}{n+1} \cdot \frac{n}{\cancel{(3x-2)^n}}\right|$$

$$= \lim_{n\to\infty} \frac{n}{n+1}\,|3x-2|$$

$$= |3x-2| < 1$$

$$-1 < 3x-2 < 1$$

$$1 < 3x < 3$$

$$\boxed{\frac{1}{3} < x < 1}$$

☆ for $x = \frac{1}{3}$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, converges conditionaly by A.S.T

($u_n = \frac{1}{n}$, 1) $u_n > 0$, 2) $u_n \searrow$ 3) $\lim_{n\to\infty} u_n = 0$).

☆ for $x=1$:> $\sum_{n=1}^{\infty} \frac{1^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$, diverges (harmonic series).

$\boxed{a}$ The radius is $\frac{1}{3}$, the Interval of convergence is $\frac{1}{3} \leq x < 1$.

$\boxed{b}$ The interval of absolute convergence is $\frac{1}{3} < x < 1$.

$\boxed{c}$ The series converges conditionally at $x = \frac{1}{3}$.

$\boxed{12}$ $\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$

Ratio test: The series converges absolutely if:

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| < 1.$$

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{3^{n+1} x^{n+1}}{(n+1)!}}{\frac{3^n x^n}{n!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{3^n} \cdot 3 \cancel{x^n} x}{(n+1) \cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{3^n} \cancel{x^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{3x}{n+1} \right|$$

$$= 3|x| \lim_{n\to\infty} \frac{1}{n+1}$$

$$= 0 < 1 \quad \text{for all } x.$$

$\boxed{a}$ the radius is ∞; the series converges for all x.

$\boxed{b}$ the series converges absolutely for all x.

$\boxed{c}$ there are no values for which the series converges conditionally.

14 $\sum_{n=1}^{\infty} \frac{(x-1)^n}{3^n n^2}$

Ratio test:

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(x-1)^{n+1}}{3^{n+1}(n+1)^2}}{\frac{(x-1)^n}{3^n n^2}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{(x-1)^n}\cdot(x-1)}{\cancel{3^n}\cdot 3\,(n+1)^2} \cdot \frac{\cancel{3^n}\, n^2}{\cancel{(x-1)^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x-1)\, n^2}{3(n+1)^2} \right|$$

$$= \frac{|x-1|}{3} \lim_{n\to\infty} \frac{n^2}{(n+1)^2}$$

$$= \frac{|x-1|}{3} \cdot 1$$

$$= \frac{|x-1|}{3}.$$

The series converges absolutely when $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1.$

$$\Rightarrow \frac{|x-1|}{3} < 1$$

$$\Rightarrow |x-1| < 3$$

$$\Rightarrow -3 < x-1 < 3$$

$$\boxed{-2 < x < 4}$$

★ When $x=-2$:- we have $\sum_{n=1}^{\infty} \frac{(-3)^n}{3^n n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, an absolutely convergent series.

★ When $x=4$:- we have $\sum_{n=1}^{\infty} \frac{3^n}{3^n n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$, an absolutely convergent series.

a) The radius is 3; the interval of convergence is $-2 \leqslant x \leqslant 4$.

b) The interval of absolute convergence is $-2 \leqslant x \leqslant 4$.

c) There are no values for which the series converges conditionally.

---

23) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n x^n$

Ratio test: The series converges absolutely when $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$.

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left| \frac{\left(1+\frac{1}{n+1}\right)^{n+1} x^{n+1}}{\left(1+\frac{1}{n}\right)^n x^n} \right|$$

$$= |x| \frac{\lim_{n\to\infty} \left(1+\frac{1}{n+1}\right)^{n+1}}{\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n}$$

$$\lim_{n\to\infty} \left(1+\frac{1}{n+1}\right)^{n+1}$$

let $L = n+1$.

$n \to \infty$, $L \to \infty$

$$\lim_{L\to\infty} \left(1+\frac{1}{L}\right)^L = e^1 = e$$

$$= |x| \cdot \frac{e}{e}$$

$$= |x|.$$

$$\Rightarrow |x| < 1.$$

$$-1 < x < 1.$$

★ When $x=-1$ we have $\sum_{n=1}^{\infty} (-1)^n \left(1+\frac{1}{n}\right)^n$:-

a divergent series by nth term test since

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \neq 0.$$

✩ when $x = 1$ we have $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n$:-

a divergent series by n-th term test.

(a) the radius is 1, the interval of convergence $-1 < x < 1$.

(b) the interval of absolute convergence is $-1 < x < 1$.

(c) there are no values for which the series converges conditionally.

---

29 $\sum_{n=2}^{\infty} \frac{x^n}{n(\ln n)^2}$

Ratio test :- The series converges absolutely when :-

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1.$$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1(\ln(n+1))^2}\cdot\frac{n(\ln n)^2}{x^n}\right| < 1.$$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{x\, n(\ln n)^2}{n+1(\ln(n+1))^2}\right| < 1.$$

$$\Rightarrow |x| \lim_{n\to\infty}\left(\frac{n}{n+1}\right)\left(\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)}\right)^2 < 1.$$

$$\Rightarrow |x|\cdot(1)\left(\lim_{n\to\infty}\frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n+1}\right)}\right)^2 < 1.$$

$$\Rightarrow |x|\left(\lim_{n\to\infty}\frac{n+1}{n}\right)^2 < 1.$$

$$\Rightarrow |x|\,(1)^2 < 1.$$

$$\Rightarrow |x| < 1$$

$$\Rightarrow \boxed{-1 < x < 1}$$

✩ when $x = -1$, $\sum_{n=2}^{\infty}\frac{(-1)^n}{n(\ln n)^2}$

which converges absolutely by the integral test.

$\Rightarrow \sum_{n=2}^{\infty}\left|\frac{(-1)^n}{n(\ln n)^2}\right| = \sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$

have positive terms

$\Rightarrow$ let $f(x) = \frac{1}{x(\ln x)^2}$, cont, positive & decreasing $\forall x \geq 2$.

$\Rightarrow \int_2^{\infty}\frac{1}{x(\ln x)^2}\,dx$, let $u = \ln x$, $du = \frac{dx}{x}$

$\Rightarrow \lim_{b\to\infty}\int_{\ln 2}^{b} u^{-2}\,du = \lim_{b\to\infty} -\frac{1}{u}\Big]_{\ln 2}^{b}$

$= \lim_{b\to\infty} -\frac{1}{b} + \frac{1}{\ln 2} = \frac{1}{\ln 2}$ (converges)

☆ when $x = 1$, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$, Converges by the integral test.

[a] the radius is 1, the interval of convergence is $-1 \leqslant x \leqslant 1$

[b] the interval of absolute convergence is $-1 \leqslant x \leqslant 1$.

[c] there are no values for which the series Converges conditionally.

---

[32] $\sum_{n=1}^{\infty} \frac{(3x+1)^{n+1}}{2n+2}$

by ratio test, the series converges absolutely if :-

$$\Rightarrow \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| < 1.$$

$$\Rightarrow \lim_{n\to\infty} \left| \frac{(3x+1)^{n+2}}{2n+4} \cdot \frac{2n+2}{(3x+1)^{n+1}} \right| < 1.$$

$$\Rightarrow \lim_{n\to\infty} \left| (3x+1) \cdot \frac{2n+2}{2n+4} \right| < 1.$$

$$\Rightarrow |3x+1| \lim_{n\to\infty} \frac{2n+2}{2n+4} < 1.$$

$$\Rightarrow |3x+1| \cdot (1) < 1.$$

$$\Rightarrow |3x+1| < 1.$$

$$\Rightarrow -1 < 3x+1 < 1$$

$$\Rightarrow -2 < 3x < 0$$

$$\Rightarrow \boxed{-\frac{2}{3} < x < 0}$$

☆ when $x = -\frac{2}{3}$, we have $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+2}$,

a conditionally convergent series by A.S.T.

☆ when $x = 0$, we have $\sum_{n=1}^{\infty} \frac{1}{2n+2}$

a divergent series.

☆ $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+2}$, $u_n = \frac{1}{2n+2}$

A.S.T :-

1) $u_n > 0$. ✓

2) $u_n > u_{n+1}$. ✓

3) $\lim_{n\to\infty} u_n = 0$ ✓

☆ $\sum_{n=1}^{\infty} \frac{1}{2n+2}$, use L.C.T & Compare with $\sum_{n=1}^{\infty} \frac{1}{n}$ (div harmonic series)

$\lim_{n\to\infty} \frac{\frac{1}{2n+2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2n+2} = \frac{1}{2} > 0$ (both div)

(a) the radius is $\frac{1}{3}$, the interval of convergence is $-\frac{2}{3} \leq x < 0$.

(b) the interval of absolute convergence is $-\frac{2}{3} < x < 0$.

(c) the series converges conditionally at $x = -\frac{2}{3}$.

---

40 Find the series' radius of convergence

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} x^n$$

Root test :- The series converges absolutely if

$$\lim_{n\to\infty} \sqrt[n]{|u_n|} < 1$$

$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n}{n+1}\right)^{n^2} x^n\right|} < 1.$$

$$\Rightarrow \lim_{n\to\infty} \left|\left(\frac{n}{n+1}\right)^{\frac{n^2}{n}} x^{\frac{n}{n}}\right| < 1.$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n} |x| < 1.$$

$$\Rightarrow |x| \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n} < 1.$$

$$\Rightarrow |x| \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} < 1.$$

$$\Rightarrow |x| \cdot \frac{1}{e^1} < 1.$$

$$\Rightarrow |x| < e$$

$$\Rightarrow -e < x < e.$$

So, The radius of convergence $R = e$.

46. Use theorem (20) – in the text book – to find the series' interval of convergence & within this interval, the sum of the series as a function

$$\sum_{n=0}^{\infty} (\ln x)^n$$

Theorem (20): If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$, then

$\sum_{n=0}^{\infty} a_n (f(x))^n$ converges absolutely for any continuous function

f on $|f(x)| < R$.

by Ratio test: $\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1$.

⇒ The series converges absolutely if

$$\Rightarrow \lim_{n \to \infty} \left| \frac{(\ln x)^{n+1}}{(\ln x)^n} \right| < 1.$$

$$\Rightarrow \lim_{n \to \infty} |\ln x| < 1$$

$$\Rightarrow |\ln x| < 1.$$

$$\Rightarrow -1 < \ln x < 1.$$

$$\Rightarrow e^{-1} < x < e$$

* when $x = e^{-1}$ we have $\sum_{n=0}^{\infty} (\ln(e^{-1}))^n = \sum_{n=1}^{\infty} (-1)^n$ which is a divergent series.

* when $x = e$ we have $\sum_{n=0}^{\infty} (\ln e)^n = \sum_{n=0}^{\infty} 1^n$ which is a divergent series.

⇒ the interval of convergence is $e^{-1} < x < e$.

⇒ the sum of the series is $\frac{1}{1 - \ln x}$ when $e^{-1} < x < e$

(since $\sum_{n=0}^{\infty} (\ln x)^n$ is a geometric series with $r = \ln x$ & $a = 1$).

49 For what values of x does the series

$1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-3)^n + \cdots$ converge? what is its sum?

what series do you get if you differentiate the given series term by term?

$\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-3)^n$ converges absolutely if $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$ (Ratio test).

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$$

$$\Rightarrow \lim_{n\to\infty} \left| \frac{\left(\frac{1}{2}\right)^{n+1} (x-3)^{n+1}}{\left(\frac{1}{2}\right)^n (x-3)^n} \right| < 1.$$

$$\Rightarrow \lim_{n\to\infty} \left|\frac{1}{2}(x-3)\right| < 1.$$

$$\Rightarrow \frac{1}{2}|x-3| < 1.$$

$$\Rightarrow |x-3| < 2.$$

$$\Rightarrow -2 < x-3 < 2$$

$$\Rightarrow \boxed{1 < x < 5}$$

when $x=1$, we have $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (-2)^n = \sum_{n=0}^{\infty} 1^n$ which diverges.

when $x=5$, we have $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (2)^n = \sum_{n=0}^{\infty} (-1)^n$ which diverges.

$\Rightarrow$ The interval of convergence is $1 < x < 5$.

$\Rightarrow$ The sum of this convergent geometric series is $\frac{1}{1+\left(\frac{x-3}{2}\right)} = \frac{1}{\frac{2+x-3}{2}} = \boxed{\frac{2}{x-1}}$

Now, $f(x) = 1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-3)^n + \cdots = \frac{2}{x-1}$

$f'(x) = 0 - \frac{1}{2} + \frac{1}{2}(x-3) + \cdots + \left(-\frac{1}{2}\right)^n n (x-3)^{n-1} + \cdots = -\frac{2}{(x-1)^2}$

is convergent when $1 < x < 5$, & diverges when $x=1$ or 5.

Note that: the sum for $f'(x)$ is $\frac{-2}{(x-1)^2}$, the derivative of $\frac{2}{x-1}$.

10.8 Taylor & Maclaurin Series

3) Find the taylor Polynomials of orders 0, 1, 2 & 3 generated by f at a.

$f(x) = \ln x \quad , \quad a = 1.$

$f(1) = \ln 1 = 0.$

$f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1.$

$f''(x) = -\frac{1}{x^2} \Rightarrow f''(1) = -1$

$f'''(x) = +\frac{2}{x^3} \Rightarrow f'''(1) = 2$

$P_0(x) = f(1) = 0$

$$\begin{aligned} P_1(x) &= f(1) + f'(1)(x-1) \\ &= 0 + 1(x-1) \\ &= x-1 \end{aligned}$$

$$\begin{aligned} P_2(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 \\ &= 0 + 1(x-1) + \frac{-1}{2}(x-1)^2 \\ &= (x-1) - \frac{1}{2}(x-1)^2 \end{aligned}$$

$$\begin{aligned} P_3(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 \\ &= 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{6}(x-1)^3 \\ &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} \end{aligned}$$

(14) Find the Maclaurin series for the function

$$f(x) = \frac{2+x}{1-x}$$

Maclaurin series generated by f is

$$f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$f(0) = 2$

$f'(x) = \frac{3}{(1-x)^2} \Rightarrow f'(0) = 3$

$f''(x) = \frac{6}{(1-x)^3} \Rightarrow f''(0) = 6$

$f'''(x) = \frac{18}{(1-x)^4} \Rightarrow f'''(0) = 18.$

$\vdots$

The Taylor series is

$$2 + 3x + \frac{6}{2!}(x^2) + \frac{18}{3!}x^3 + \dots$$

$$= 2 + 3x + 3x^2 + 3x^3 + \dots$$

$$= 2 + \sum_{n=1}^{\infty} 3x^n.$$

20 Find the Maclaurin series for the function

$f(x) = \sinh x = \frac{e^x - e^{-x}}{2}$.

$f(0) = \sinh(0) = \frac{e^0 - e^{-0}}{2} = 0$.

$f'(x) = \frac{e^x + e^{-x}}{2} \Rightarrow f'(0) = \frac{1+1}{2} = 1$

$f''(x) = \frac{e^x - e^{-x}}{2} \Rightarrow f''(0) = 0$

$f'''(x) = \frac{e^x + e^{-x}}{2} \Rightarrow f'''(0) = 1$

$\vdots$

Maclaurin series $\Rightarrow f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$

$\sinh x = 0 + 1(x) + \frac{0\,(x^2)}{2} + \frac{1}{3!}x^3 + \cdots$

$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

22) Find the Maclaurin series of the function

$$f(x) = \frac{x^2}{x+1}$$

$f(0) = 0$

$f'(x) = \frac{x^2+2x}{(x+1)^2} \Rightarrow f'(0) = 0$

$f''(x) = \frac{2}{(x+1)^3} \Rightarrow f''(0) = 2.$

$f'''(x) = \frac{-6}{(x+1)^4} \Rightarrow f'''(0) = -6$

$f^{(n)}(x) = \frac{(-1)^n n!}{(x+1)^{n+1}} \Rightarrow f^{(n)}(0) = (-1)^n n!$, If $n \geq 2$

Maclaurin series of $f = 0 + 0 + \frac{2}{2!} x^2 + \frac{-6}{3!} x^3 + \cdots$

$= x^2 - x^3 + x^4 - x^5 + \cdots$

$= \sum_{n=2}^{\infty} (-1)^n x^n.$

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27) Find the Taylor series generated by f at $x=a$.

$f(x) = \frac{1}{x^2}$, $a=1$.

$f(1) = 1$

$f'(x) = -\frac{2}{x^3} \Rightarrow f'(1) = -2$

$f''(x) = \frac{6}{x^4} = \frac{3!}{x^4} \Rightarrow f''(1) = 3!$

$f'''(x) = -\frac{4!}{x^5} \Rightarrow f'''(1) = -4!$

$f^n(x) = \frac{(-1)^n (n+1)!}{x^{n+2}} \Rightarrow f^n(1) = (-1)^n (n+1)!$

Taylor series is $f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \cdots$

$= 1 - 2(x-1) + \frac{3!}{2!}(x-1)^2 - \frac{4!}{3!}(x-1)^3 + \cdots$

$= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + \cdots$

$= \sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n$

32 Find the Taylor series generated by

$$f(x) = \sqrt{x+1} \quad , \quad a = 0$$

$$f(0) = 1$$

$$f'(x) = \frac{1}{2\sqrt{x+1}} \Rightarrow f'(0) = \frac{1}{2}.$$

$$f''(x) = \frac{-1}{4(x+1)^{3/2}} \Rightarrow f''(0) = -\frac{1}{4}.$$

$$f'''(x) = \frac{3}{8(x+1)^{5/2}} \Rightarrow f'''(0) = \frac{3}{8}$$

$$f^{(4)}(x) = \frac{-15}{16(x+1)^{7/2}} \Rightarrow f^{(4)}(1) = \frac{-15}{16}$$

$$\vdots$$

Taylor series: $f(0) + f'(0)(x-0) + \frac{f''(0)(x-0)^2}{2!} + \frac{f'''(0)(x-0)^3}{3!} + \cdots$

$$\sqrt{x+1} = 1 + \frac{1}{2}x + -\frac{1}{4(2!)}x^2 + \frac{3}{8(3!)}x^3 + \frac{-15}{16(4!)}x^4 + \cdots$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5}{128}x^4 + \cdots$$

37 Use the Taylor series generated by e^x at $x=a$ to show that:

$$e^x = e^a\left[1+(x-a)+\frac{(x-a)^2}{2!}+\cdots\right].$$

$f(a) = e^a$

$f'(x) = e^x \Rightarrow f'(a) = e^a$.

$f''(x) = e^x \Rightarrow f''(a) = e^a$

$f'''(x) = e^x \Rightarrow f'''(a) = e^a$.

$\vdots$

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

* $f(x) = e^x$ & * $f^n(a) = e^a$ for all $n = 0, 1, 2, 3, \ldots$

$$e^x = \frac{e^a (x-a)^0}{0!} + \frac{e^a (x-a)}{1!} + \frac{e^a (x-a)^2}{2!} + \frac{e^a (x-a)^3}{3!} + \cdots$$

$$= e^a\left[1+(x-a)+\frac{(x-a)^2}{2!}+\frac{(x-a)^3}{3!}+\cdots\right]. \text{ at } x=a$$

10.9 Convergence of Taylor series.

10 Use substitution to find the Taylor series at $x = 0$ of the function $\frac{1}{2-x}$.

$$\frac{1}{2-x} = \frac{1}{2\left(1-\frac{x}{2}\right)}$$

$$= \frac{1}{2}\left[\frac{1}{1-\frac{x}{2}}\right].$$

Taylor series at $a=0$ of $\frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$

Notice that:- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ {geometric series with $a=1$ & $r=x$

So, by substitution we have:-

$$\frac{1}{1-\frac{x}{2}} = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$$

Now, $\frac{1}{2-x} = \frac{1}{2}\left[\frac{1}{1-\frac{x}{2}}\right]$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}$$

$$= \frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \cdots .$$

$\boxed{12}$ Use power series operations to find the Taylor series at $x=0$ for the function $\boxed{x^2 \sin x}$

Remember:

Taylor series generated by $\boxed{\sin x}$ at $x=0$ is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots .$$

$$\boxed{x^2 \sin x} = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+1)!}$$

$$= x^3 - \frac{x^5}{3!} + \frac{x^7}{5!} - \frac{x^9}{7!} + \cdots .$$

18) Use power series operations to find the Taylor series at $x=0$ for the function $\sin^2 x$.

we know that :-

★ Remember:-

$\cos(2x) = 1 - 2\sin^2 x$.

$$\boxed{\sin^2 x = \frac{1}{2} - \frac{\cos(2x)}{2}}$$

$$\Rightarrow \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\Rightarrow \cos(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} x^{2n}}{(2n)!}$$

$$\Rightarrow -\frac{1}{2}\cos(2x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{2^{2n-1} x^{2n}}{(2n)!} = -\frac{1}{2} + \frac{2x^2}{2!} - \frac{2^3 x^4}{4!} + \cdots$$

$$\Rightarrow \boxed{\sin^2 x} = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

$$= \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} + \frac{2x^2}{2!} - \frac{2^3 x^4}{4!} + \frac{2^5 x^6}{6!} - \cdots$$

$$= \boxed{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^{2n-1} x^{2n}}{(2n)!}}$$

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28 Use power series operations to find the Taylor series at $x=0$ for the function $\ln(x+1) - \ln(1-x)$.

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n$$

Remember $\Rightarrow$ $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.

$$\Rightarrow \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$= 1 - x + x^2 - x^3 + \cdots$$

$$\int \frac{1}{1+x}\,dx = \int (1 - x + x^2 - x^3 + \cdots)\,dx.$$

$$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots} \quad \text{☆}$$

$$\Rightarrow \& \quad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

$$\int \frac{1}{1-x}\,dx = \int (1 + x + x^2 + x^3 + \cdots)\,dx.$$

$$\boxed{-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots} \quad \text{☆☆}$$

Now, ☆ + ☆☆

$$\boxed{\ln(1+x) - \ln(1-x)} = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \cdots$$

$$\boxed{= \sum_{n=0}^{\infty} 2\,\frac{x^{2n+1}}{2n+1}}$$

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35) Estimate the error if $P_3(x) = x - \frac{x^3}{6}$ is used to estimate the value of $\sin x$ at $x = 0.1$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$P_3(x) = x - \frac{x^3}{3!} = x - \frac{x^3}{6}$$

$f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x.$

$f'''(x) = -\cos x.$

$f^{(4)}(x) = \sin x$

Since $n = 3 \Rightarrow f^{(4)}(x) = \sin x.$

$|f^{(4)}(x)| \leq M$ on $[0, 0.1]$

$|\sin x| \leq 1$ on $[0, 0.1]$

$\Rightarrow$ So, $M = 1$

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$$|R_3(0.1)| \leqslant \frac{M\,|0.1-0|^{3+1}}{(3+1)\,!}$$

$$|R_3(0.1)| \leqslant \frac{(1)\,|0.1|^4}{4!}$$

$$\leqslant \frac{0.1^4}{4!} \approx 4.167 \times 10^{-6}$$

Note: You can use alternating series estimation theorem to solve this question.

37) For approximately what values of x can you replace $\sin x$ by $x - \frac{x^3}{6}$ with an error of magnitude no greater than 5×10^{-4} ?

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

By alternating series estimation theorem:

$\Rightarrow$ $|\text{errer}| < |\text{first neglected term}|$, then

$$|\text{error}| < \frac{|x^5|}{5!}, \text{ so:}$$

$$\frac{|x^5|}{5!} < 5\times10^{-4} \Rightarrow |x|^5 < 5!\,(5\times10^{-4}) = 0.06$$

$$\Rightarrow |x| < \sqrt[5]{0.06} \approx 0.5698$$

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41) The approximation $e^x = 1 + x + \frac{x^2}{2}$ is used when x is small. Use the Reminder Estimation Theorem to estimate the error when $|x| < 0.1$.

$$e^x = \underbrace{1 + x + \frac{x^2}{2}}_{P_2(x)}$$ (Taylor Polynomial of order 2. $n=2$)

$$f^{(n+1)}(x) = f^{3}(x) = e^x,$$

but $|x| < 0.1 \Rightarrow |f^{(3)}(x)| < |e^x| = e^x < e^{0.1} < 3^{0.1}$

Hence, By remainder estimation theorem

$$|R_2(x)| < \frac{3^{0.1}|x|^3}{3!}$$

$$< \frac{3^{0.1}(0.1)^3}{3!} = 1.87 \times 10^{-4}$$

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10.10 : The Binomial series & Applications of Taylor Series.

10 Find the first four terms of the binomial series for the function $\dfrac{x}{\sqrt[3]{1+x}}$

remember:

The Binomial series for

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k$$

$$\binom{m}{k} = \frac{m(m-1)(m-2)\dots(m-k+1)}{k!}$$

$$\frac{x}{\sqrt[3]{1+x}} = x(1+x)^{-\frac{1}{3}} \quad , m = -\frac{1}{3}$$

$$= x\left[1 + \sum_{k=1}^{\infty} \binom{-\frac{1}{3}}{k} x^k\right]$$

$$= x\left[1 + \binom{-\frac{1}{3}}{1} x^1 + \binom{-\frac{1}{3}}{2} x^2 + \binom{-\frac{1}{3}}{3} x^3 + \binom{-\frac{1}{3}}{4} x^4 + \dots\right]$$

$$= x\left[1 + -\frac{1}{3}x + \frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{2} x^2 + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{1}{3}-3+1\right)}{3!} x^3 + \dots\right]$$

$$= x - \frac{x^2}{3} + \frac{4}{9(2)} x^3 + \frac{\left(\frac{4}{9}\right)\left(-\frac{7}{3}\right)}{6} x^4 + \dots]$$

$$\frac{x}{\sqrt[3]{1+x}} = x - \frac{x^2}{3} + \frac{2}{9}x^3 - \frac{14}{81} x^4 + \dots .$$

16) Use the series to estimate the integral's values with an error of magnitude less than 10^{-3}.

$$\int_0^{0.2} \frac{e^{-x}-1}{x}\,dx.$$

we know, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\Rightarrow e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

$$\Rightarrow \frac{e^{-x}-1}{x} = \frac{1}{x}\left(e^{-x}-1\right) = \frac{1}{x}\left(\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!} - 1\right).$$

$$= \frac{1}{x}\left(\left(\cancel{1} - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right) - \cancel{1}\right)$$

$$= \frac{1}{x}\left(-x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right)$$

$$= -1 + \frac{x}{2!} - \frac{x^2}{3!} + \frac{x^3}{4!} - \cdots$$

$$\Rightarrow \int_0^{0.2} \frac{e^{-x}-1}{x}\,dx = \int_0^{0.2}\left(-1 + \frac{x}{2!} - \frac{x^2}{3!} + \frac{x^3}{4!} - \cdots\right)dx.$$

$$= -x + \frac{x^2}{2.2!} - \frac{x^3}{3.3!} + \frac{x^4}{4.4!} - \cdots \Big|_0^{0.2}$$

$$= -0.2 + \frac{(0.2)^2}{2.2!} - \frac{(0.2)^3}{3.3!} + \frac{(0.2)^4}{4.4!} - \cdots$$

Now, with an error of magnitude less than 10^{-3} ($|error| < 10^{-3}$), we find the first term to be numerically less than 10^{-3}:

$$\frac{(0.2)^3}{3.3!} = \frac{0.008}{18} = 4.\bar{4} \times 10^{-4} < 10^{-3}$$

So, the first neglected term is $\frac{(0.2)^3}{3 \cdot 3!}$.

$$\int_0^{0.2} \left(\frac{e^{-x} - 1}{x} \right) dx \approx -(0.2) + \frac{(0.2)^2}{4}$$ [The estimated value of the integral]

(26) Find a polynomial that will approximate $F(x)$ throughout the given interval with an error of magnitude less than 10^{-3}.

$$F(x) = \int_0^x t^2 e^{-t^2} \, dt \,, \quad [0, 1].$$

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \Rightarrow e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}$$

$$t^2 e^{-t^2} = t^2 \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+2}}{n!}$$

$$t^2 e^{-t^2} = t^2 - t^4 + \frac{t^6}{2!} - \frac{t^8}{3!} + \frac{t^{10}}{4!} - \frac{t^{12}}{5!} + \cdots$$

$$\Rightarrow F(x) = \int_0^x t^2 e^{-t^2} dt = \int_0^x \left(t^2 - t^4 + \frac{t^6}{2!} - \frac{t^8}{3!} + \frac{t^{10}}{4!} - \frac{t^{12}}{5!} + \cdots \right) dt$$

$$= \frac{t^3}{3} - \frac{t^5}{5} + \frac{t^7}{7 \cdot 2!} - \frac{t^9}{9 \cdot 3!} + \frac{t^{11}}{11 \cdot 4!} - \frac{t^{13}}{13 \cdot 5!} + \cdots \Big|_0^x$$

$$= \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7 \cdot 2!} - \frac{x^9}{9 \cdot 3!} + \frac{x^{11}}{11 \cdot 4!} - \frac{x^{13}}{13 \cdot 5!} + \cdots$$

$\Rightarrow$ Now, with an error of magnitude less than 10^{-3}, we find that the first term to be numerically less than 10^{-3} (0.001) is $\frac{1}{13 \cdot 5!}$

$$|\text{error}| < \left| \frac{x^{13}}{13 \cdot 5!} \right| = \frac{|x|^{13}}{13 \cdot 5!} < \frac{1}{13 \cdot 5!} \approx 6.4 \times 10^{-4} < 10^{-3}.$$

$$\Rightarrow \text{So, } F(x) \approx \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7 \cdot 2!} - \frac{x^9}{9 \cdot 3!} + \frac{x^{11}}{11 \cdot 4!}.$$

Use series to evaluate the Limits :-

30) $\lim_{x \to 0} \frac{e^x - e^{-x}}{x}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$e^x - e^{-x} = 2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \cdots$$

$$\frac{e^x - e^{-x}}{x} = 2 + \frac{2x^2}{3!} + \frac{2x^4}{5!} + \cdots$$

$$\lim_{x \to 0} \frac{e^x - e^{-x}}{x} = \lim_{x \to 0} \left(2 + \frac{2x^2}{3!} + \frac{2x^4}{5!} + \cdots\right).$$

$$= \boxed{2}$$

33) $\lim_{y \to 0} \frac{y - \tan^{-1} y}{y^3}$

$$\tan^{-1} y = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{2n+1}, \quad |y| \leq 1.$$

$$= y - \frac{y^3}{3} + \frac{y^5}{5} - \frac{y^7}{7} + \cdots$$

$$\lim_{y \to 0} \frac{y - \tan^{-1} y}{y^3} = \lim_{y \to 0} \frac{\cancel{y} - \cancel{y} + \frac{y^3}{3} - \frac{y^5}{5} + \frac{y^7}{7} - \cdots}{y^3}$$

$$= \lim_{y \to 0} \left(\frac{1}{3} - \frac{y^2}{5} + \frac{y^4}{7} - \cdots\right)$$

$$= \boxed{\frac{1}{3}}$$

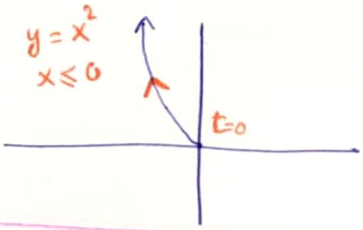
11.1 Prametrization of Plane curves.

Identify the particl's path by finding a Cartesian equation. Graph the Carlisian equation. Indicate the portion of the graph traced by the particle & the direction of motion.

2 $x = -\sqrt{t}$, $y = t$, $t \geq 0$

$x = -\sqrt{y}$ or $y = x^2$, $x \leq 0$, $y \geq 0$

(Initial Point) IP : $t=0$
$\Rightarrow (x,y) = (0,0)$.

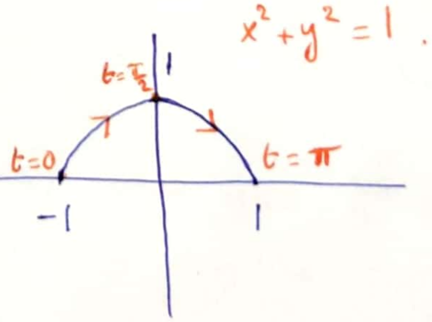


6 $x = \cos(\pi - t)$, $y = \sin(\pi - t)$, $0 \leq t \leq \pi$

$x^2 + y^2 = \cos^2(\pi - t) + \sin^2(\pi - t) = 1$

$\Rightarrow x^2 + y^2 = 1$

IP : when $t=0$
$\Rightarrow (x,y) = (\cos(\pi), \sin(\pi))$
$= (-1,0)$.



TP : when $t = \pi$.
$\Rightarrow (x,y) = (\cos 0, \sin 0)$
$= (1,0)$

To check the direction :
$\Rightarrow t = \frac{\pi}{2}$
$\Rightarrow (x,y) = (\cos\frac{\pi}{2}, \sin\frac{\pi}{2})$
$= (0,1)$

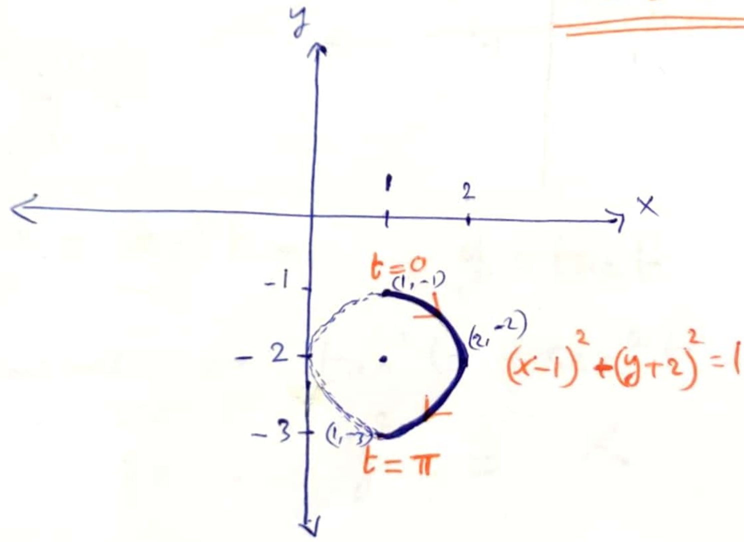
$\boxed{10}$ $x = 1+\sin t$, $y = \cos t - 2$, $0 \leq t \leq \pi$

$x = 1 + \sin t \Rightarrow \sin t = x - 1 \Rightarrow \boxed{\sin^2 t = (x-1)^2}$

$y = \cos t - 2 \Rightarrow \cos t = y + 2 \Rightarrow \boxed{\cos^2 t = (y+2)^2}$

$\sin^2 t + \cos^2 t = (x-1)^2 + (y+2)^2 = 1$

$(x-1)^2 + (y+2)^2 = 1$ *(it is a circle equation with center $(1,-2)$ & a radius of 1).



Note that → we know

$-1 \leq \cos t \leq 1.$

$-3 \leq \cos t - 2 \leq -1$

$-3 \leq y \leq -1$

So, $\boxed{y < 0}$

& $-1 \leq \sin t \leq 1.$

$0 \leq 1 + \sin t \leq 2.$

$0 \leq x \leq 2.$

So, $\boxed{x \geq 0}$.

★ IP : $t = 0$

$(x,y) = (1+\sin(0), \cos(0) - 2)$
$= (1, -1)$

when $y<0$ & $x \geq 0$
$\Rightarrow$ the curve in the fourth quadrent.

★ TP: $t = \pi$

$(x,y) = (1 + \sin(\pi), \cos(\pi) - 2)$
$= (1+0, -1-2) = (1,-3)$

★ $t = \frac{\pi}{2} \Rightarrow (x,y) = (2, -2)$

14) $x = \sqrt{t+1}$, $y = \sqrt{t}$, $t \geq 0$

$y = \sqrt{t} \Rightarrow y^2 = t$

$\Rightarrow x = \sqrt{y^2+1}$, $y \geq 0$.

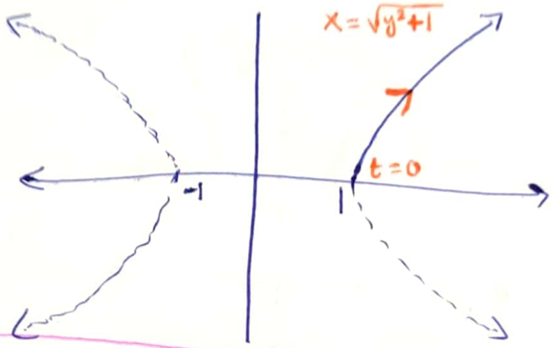
$\Rightarrow x^2 = y^2 + 1$

$\Rightarrow x^2 - y^2 = 1$

it's a hyperbola equation with $y \geq 0$ & $x \geq 1$

* when $t=0$
  IP $(x, y) = (1, 0)$

* No TP.



$t=0$ $(1, 0)$

$t=1$ $(\sqrt{2}, 1)$

15) $x = \sec^2 t - 1$, $y = \tan t$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

remember: $\tan^2 t = \sec^2 t - 1$

so, $y^2 = x$

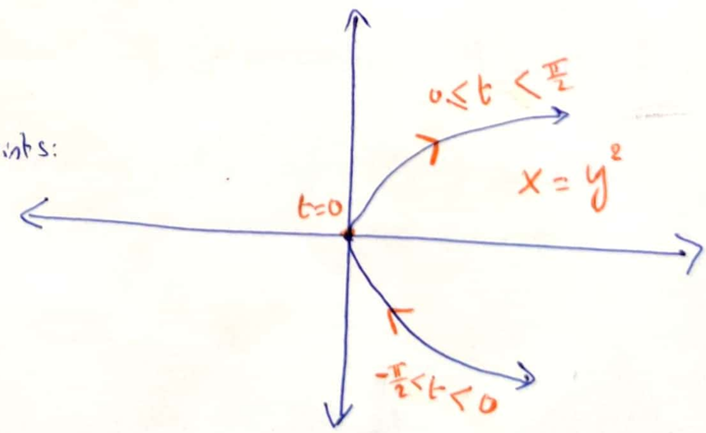
* No IP & No TP.

To check the direction, take two points:

* when $t = 0$
  $\Rightarrow (x, y) = (0, 0)$

* when $t = \frac{\pi}{4}$
  $\Rightarrow (x, y) = (1, 1)$

since $x = y^2 \Rightarrow x \geq 0$.



18 $x = 2\sinh t$, $y = 2\cosh t$, $-\infty < t < \infty$.

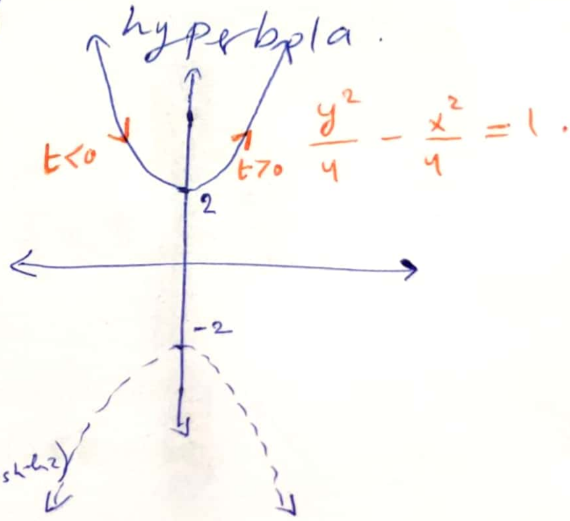
$\sinh t = \frac{x}{2}$

$\cosh t = \frac{y}{2}$, $y > 0$ $(y \geq 2)$.

$\cosh^2 t - \sinh^2 t = 1$.

$\left(\frac{y}{2}\right)^2 - \left(\frac{x}{2}\right)^2 = 1$.

$\frac{y^2}{4} - \frac{x^2}{4} = 1$ hyperbola.



☆ No IP & No TP.

when $t = 0$
$\Rightarrow (x, y) = (0, 2)$

☆ when $t = \ln 2$
$\Rightarrow (x, y) = (2\sinh \ln 2, 2\cosh \ln 2)$
$= \left(\frac{3}{2}, \frac{5}{2}\right)$.

---

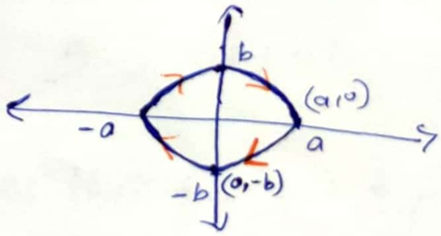
20 Find Parametric equations & parametric interval for the motion of a particle that starts at $(a, 0)$ & traces the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

a) Once clockwise

$\frac{x}{a} = \sin t \Rightarrow x = a\sin t$

$\frac{y}{b} = \cos t \Rightarrow y = b\cos t$.

$\frac{\pi}{2} \leq t \leq \frac{5\pi}{2}$



☆ IP when $t = \frac{\pi}{2} \Rightarrow (x, y) = (a, 0)$

☆ TP when $t = \frac{5\pi}{2} \Rightarrow (x, y) = (a, 0)$

☆ To check the direction:
take $t = \pi$.
$x = a\sin\pi = 0$.
$y = b\cos\pi = -b$
$\Big\} (0, -b)$

(b) Once Counterclockwise.

$\frac{x}{a} = \cos t \Rightarrow x = a \cos t.$

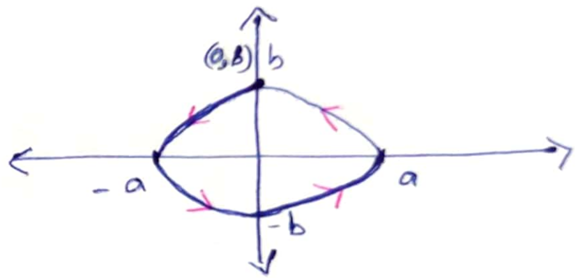
$\frac{y}{b} = \sin t \Rightarrow y = b \sin t.$ $\qquad 0 \leq t \leq 2\pi$

IP: $t = 0 \Rightarrow (x, y) = (a, 0)$

TP: $t = 2\pi \Rightarrow (x, y) = (a, 0)$

To check the direction, take $t = \frac{\pi}{2}$.

$x = a \cos\frac{\pi}{2} = 0$
$y = b \sin\frac{\pi}{2} = b$ $\Big\} (0, b)$



(c) Twice clockwise.

$x = a \sin t$

$y = b \cos t$ $\qquad \frac{\pi}{2} \leq t \leq \frac{9\pi}{2}$

d) Twice counterclockwise.

$x = a \cos t.$

$y = a \sin t$ $\qquad 0 \leq t \leq 4\pi.$

22) Find a parametrization of the curve:

The line segment with end points $(-1,3)$ & $(3,-2)$.
(x_1,y_1) (x_2,y_2)

$$m_{\text{slope}} = \frac{-2-3}{3--1} = -\frac{5}{4}$$

$$y - y_1 = m(x - x_1).$$

$$y - 3 = -\frac{5}{4}(x+1) \Rightarrow y = -\frac{5}{4}x + \frac{7}{4}.$$

Let $x = t$

so, $y = -\frac{5}{4}t + \frac{7}{4}$, $-1 \leq t \leq 3$.

another possible:

Let $t = \frac{x+1}{4}$, when $x=-1 \Rightarrow t=0$
$x=3 \Rightarrow t=1$

$\Rightarrow x = 4t - 1$
$y = -5t + 3$, $0 \leq t \leq 1$.

or

Let $t = x+1$

$x = t - 1$
$y = -\frac{5}{4}t + 3$, $0 \leq t \leq 4$.

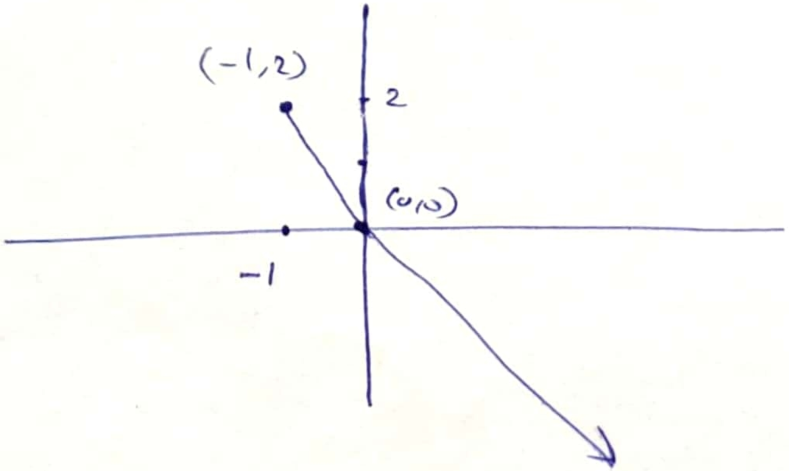
26) Find a parametrization for the ray with initial point $(-1, 2)$ that passes through the point $(0,0)$.

$$m = \frac{2-0}{-1-0} = -2.$$

$$y - 0 = -2(x-0) \Rightarrow y = -2x.$$

let $x = t$ $\quad t \geq -1$

$\Rightarrow y = -2t$



11.2 :- Calculus with Parametric Curves.

7) If $x = \sec t$, $y = \tan t$, $t = \frac{\pi}{6}$

Find the tangent line & $\frac{d^2y}{dx^2}$.

when $t = \frac{\pi}{6}$: $x = \sec\frac{\pi}{6} = \frac{2}{\sqrt{3}}$, $y = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ } The point is $\left(\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

slope $= \frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sec^2 t}{\cancel{\sec t}\tan t} = \frac{\sec t}{\tan t}.$$

$$= \frac{1}{\cancel{\cos t}} \cdot \frac{\cancel{\cos t}}{\sin t}$$

$$= \csc t.$$

So, $\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{6}} = \csc\frac{\pi}{6} = 2.$

The tangent line is :-

$$y - \frac{1}{\sqrt{3}} = 2\left(x - \frac{2}{\sqrt{3}}\right)$$

$$\boxed{y = 2x - \sqrt{3}}$$

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$$\Rightarrow \frac{d^2y}{dx^2} = \left.\frac{\frac{dy'}{dt}}{\frac{dx}{dt}}\right|_{t=\frac{\pi}{6}} = \left.\frac{-\csc t \cot t}{\sec t \tan t}\right|_{t=\frac{\pi}{6}}.$$

$$= \frac{-2 \cdot \sqrt{3}}{\frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}}$$

$$= \frac{-2\sqrt{3}}{\frac{2}{3}} = -2\sqrt{3} \cdot \frac{3}{2} = \boxed{-3\sqrt{3}}$$

T4) Find the equation of the line tangent to the curve $x = t + e^t$, $y = 1 - e^t$, $t = 0$. & Find $\frac{d^2y}{dx^2}$ at this point.

at $t=0 \Rightarrow$ $x = 0 + e^0 = 1$, $y = 1 - e^0 = 1 - 1 = 0$ $\Big\}$ (x, y) $(1, 0)$

$$\frac{dy}{dx} = \left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=0} = \left.\frac{-e^t}{1+e^t}\right|_{t=0} = \frac{-1}{1+1} = \frac{-1}{2}.$$

So, The line is :- $\boxed{y = -\frac{1}{2}x + \frac{1}{2}}$.

$$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$$

$$\frac{dy'}{dt} = \frac{(1+e^t)(-e^t) + e^t e^t}{(1+e^t)^2}$$

$$= \frac{-e^t - \cancel{e^{2t}} + \cancel{e^{2t}}}{(1+e^t)^2} = \frac{-e^t}{(1+e^t)^2}$$

So, $\left.\frac{d^2y}{dx^2}\right|_{t=0} = \left.\frac{\frac{-e^t}{(1+e^t)^2}}{1+e^t}\right|_{t=0} = \left.\frac{-e^t}{(1+e^t)^3}\right|_{t=0}$

$$= \frac{-1}{(1+1)^3} = \boxed{\frac{-1}{8}}$$

20) Find the slope of the curve $x = f(t)$, $y = g(t)$ at $t=0$ & the tangent line

$t = \ln(x-t)$ & $y = te^t$.

To find $\frac{dx}{dt} \Rightarrow$ implicit derivative

$$\Rightarrow 1 = \frac{1}{x-t}\left(\frac{dx}{dt} - 1\right).$$

$$x - t = \frac{dx}{dt} - 1$$

$$\frac{dx}{dt} = x - t + 1$$

but :- when $t=0 \Rightarrow 0 = \ln(x-0) \Rightarrow x = e^0 = 1$

$y = 0e^0 = 0$ $\Big\}$ (x, y) $(1, 0)$

So, $\frac{dx}{dt} = 1 - t + 1 = \boxed{2-t}$

$$\frac{dy}{dt} = \boxed{te^{t} + e^{t}}$$

$$\left.\frac{dy}{dx}\right|_{t=0} = \left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=0} = \left.\frac{te^{t}+e^{t}}{2-t}\right|_{t=0}$$

$$= \frac{0+1}{2-0} = \boxed{\frac{1}{2}}.$$

So, the tangent line is

$$\boxed{y = \frac{1}{2}x - \frac{1}{2}}.$$

$\boxed{22}$ Find the area enclosed by the y-axis & the curve $x = t - t^2$, $y = 1 + e^{-t}$.

$$\int x\, dy.$$

$x = t - t^2.$

$dy = -e^{-t}\, dt.$

$$\int_0^1 \left|(t-t^2)\cdot -e^{-t}\, dt\right|$$

$$= \int_0^1 \left|(t^2-t)e^{-t}\, dt\right|$$

$$= \left|\left(-e^{-t}(t^2-t) - e^{-t}(2t-1) - 2e^{-t}\right)\Big|_0^1\right|$$

$$= \left|-e^{-1}(0) - e^{-1}(-1) - 2e^{-1} - (0 - e^{0}(-1) - 2(e^{0}))\right|$$

$$= \left|-\frac{3}{e} + 1\right|$$

y-axis $x = 0$

& $x = t - t^2$

Find the intersection points.

$t - t^2 = 0$

$t(1-t) = 0$

$\boxed{t = 0, 1}$

$t^2 - t$	+	e^{-t}
$2t-1$	−	$-e^{-t}$
2	+	e^{-t}
		$-e^{-t}$

25) Find the length of the curve $x = \cos t$, $y = t + \sin t$. $0 \leq t \leq \pi$

$$L = \int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

$$\Rightarrow \frac{dx}{dt} = -\sin t \quad \Rightarrow \left(\frac{dx}{dt}\right)^2 = (-\sin t)^2 = \sin^2 t.$$

$$\Rightarrow \frac{dy}{dt} = 1 + \cos t \quad \Rightarrow \left(\frac{dy}{dt}\right)^2 = (1+\cos t)^2 = 1 + 2\cos t + \cos^2 t.$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \sin^2 t + 1 + 2\cos t + \cos^2 t.$$

$$= 1 + 1 + 2\cos t.$$

$$= 2 + 2\cos t.$$

$$= 2(1+\cos t).$$

$$\Rightarrow L = \int_0^{\pi} \sqrt{2(1+\cos t)}\, dt = \sqrt{2}\int_0^{\pi} \sqrt{1+\cos t\left(\frac{1-\cos t}{1-\cos t}\right)}\, dt.$$

$$= \sqrt{2}\int_0^{\pi} \sqrt{\frac{1-\cos^2 t}{1-\cos t}}\, dt.$$

$$= \sqrt{2}\int_0^{\pi} \frac{|\sin t|}{\sqrt{1-\cos t}}\, dt.$$

$$= \sqrt{2}\int_0^{\pi} \frac{\sin t}{\sqrt{1-\cos t}}\, dt.$$

Since $0 \leq \sin t \leq 1$ when $0 \leq t \leq \pi$

let $u = 1-\cos t$

$du = \sin t\, dt$

$$= \sqrt{2}\int \frac{du}{\sqrt{u}}$$

$$= \sqrt{2} \cdot 2\sqrt{u}$$

$$= \sqrt{2}\; 2\sqrt{1-\cos t}\,\Big|_0^{\pi}$$

$$= 2\sqrt{2}(\sqrt{2} - 0) = \boxed{4}$$

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$\boxed{27}$ Find the length of the curve

$x = \frac{t^2}{2}$, $y = \frac{(2t+1)^{3/2}}{3}$, $0 \leq t \leq 4$.

$\Rightarrow \frac{dx}{dt} = \frac{2t}{2} = t \Rightarrow \left(\frac{dx}{dt}\right)^2 = t^2.$

$\Rightarrow \frac{dy}{dt} = \frac{\frac{3}{2}(2t+1)^{\frac{1}{2}}}{3} \cdot 2 = \frac{\cancel{3} \cdot 2 (2t+1)^{\frac{1}{2}}}{\cancel{3} \cdot 2}$

$= (2t+1)^{\frac{1}{2}}$

$\left(\frac{dy}{dt}\right)^2 = \left((2t+1)^{\frac{1}{2}}\right)^2 = |2t+1| = 2t+1$ $\quad 0 \leq t \leq 4.$

$\Rightarrow \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = t^2 + 2t + 1$

$= (t+1)^2.$

$L = \int_0^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \int_0^4 \sqrt{(t+1)^2}\, dt.$

$= \int_0^4 (t+1)\, dt$

$= \left(\frac{t^2}{2} + t\right)\Big|_0^4$

$= \frac{16}{2} + 4 - 0$

$= 8 + 4$

$= \boxed{12}$

11.3 : Polar Coordinates.

1) which polar Coordinate pairs lable the same point.

a) $(3,0)$ b) $(-3,0)$ c) $\left(2,\frac{2\pi}{3}\right)$ d) $\left(2,\frac{7\pi}{3}\right)$

e) $(-3,\pi)$ f) $\left(2,\frac{\pi}{3}\right)$ g) $(-3,2\pi)$ h) $\left(-2,-\frac{\pi}{3}\right)$

Remember :- $(r,\theta) = (r,\ \theta + 2\pi n)$ ✿

$= (-r,\ \theta + (2n+1)\pi)$ ✿✿ $n = 0, \mp 1, \mp 2, \mp 3, \dots$

a) = e)

$(3,0) = (-3,\pi)$ (Compare with ✿✿ with $n=0$)

b) = g)

$(-3,0) = (-3,2\pi)$ (✿ with $n=1$).

c) = h)

$\left(2,\frac{2\pi}{3}\right) = \left(-2,-\frac{\pi}{3}\right)$ (✿✿ with $n=-1$)

d) = f)

$\left(2,\frac{\pi}{3}\right) = \left(2,\frac{7\pi}{3}\right)$ (✿ with $n=1$).

6) Find the Cartesian coordinates of the following points :-

$$\boxed{(r,\theta) \longrightarrow (x,y) \qquad x = r\cos\theta \quad y = r\sin\theta}$$

a) $\left(\sqrt{2}, \frac{\pi}{4}\right) \Rightarrow x = \sqrt{2}\cos\frac{\pi}{4} = \sqrt{2}\cdot\frac{1}{\sqrt{2}} = 1$

$y = \sqrt{2}\sin\frac{\pi}{4} = \sqrt{2}\cdot\frac{1}{\sqrt{2}} = 1$

$(x,y) = (1,1)$

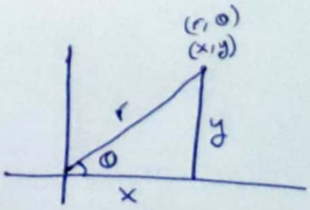
b) $(1,0) \Rightarrow \left.\begin{array}{l} x = r\cos 0 = 1 \\ y = 1\sin 0 = 0 \end{array}\right\} (x,y) = (1,0)$

c) $\left(0, \frac{\pi}{2}\right) \Rightarrow \left.\begin{array}{l} x = 0\cos\frac{\pi}{2} = 0 \\ y = 0\sin\frac{\pi}{2} = 0 \end{array}\right\} (x,y) = (0,0)$

d) $\left(-\sqrt{2}, \frac{\pi}{4}\right) \Rightarrow \left.\begin{array}{l} x = -\sqrt{2}\cos\frac{\pi}{4} = -1 \\ y = -\sqrt{2}\sin\frac{\pi}{4} = -1 \end{array}\right\} (x,y) = (-1,-1)$

e) $\left(-3, \frac{5\pi}{6}\right) \Rightarrow \left.\begin{array}{l} x = -3\cos\left(\frac{5\pi}{6}\right) = -3\left(-\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{2} \\ y = -3\sin\left(\frac{5\pi}{6}\right) = -3\cdot\frac{1}{2} = -\frac{3}{2} \end{array}\right\} (x,y) = \left(\frac{3\sqrt{3}}{2}, -\frac{3}{2}\right)$

f) $\left(5, \tan^{-1}\left(\frac{4}{3}\right)\right) \Rightarrow \left.\begin{array}{l} x = 3 \\ y = 4 \end{array}\right\} (x,y) = (3,4)$



$\tan\theta = \frac{y}{x}$

$\tan^{-1}\left(\frac{y}{x}\right) = \theta$

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g) $(-1, 7\pi) \Rightarrow x = -1\cos(7\pi) = (-1)(-1) = 1$
$y = -1\sin(7\pi) = -1(0) = 0$
$(x,y) = (1,0)$

7) Find the polar coordinates, $0 \leq \theta \leq 2\pi$ & $r \geq 0$.

$(x,y) \longrightarrow (r,\theta)$: $r^2 = x^2 + y^2$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\cos\theta = \frac{x}{r}$ & $\sin\theta = \frac{y}{r}$

a) $(1,1) = (x,y) \Rightarrow r^2 = x^2 + y^2 = 1^2 + 1^2 = 2 \Rightarrow r = \sqrt{2}$

$\cos\theta = \frac{1}{\sqrt{2}}$
$\sin\theta = \frac{1}{\sqrt{2}}$
$\theta = \frac{\pi}{4}$.

$(r,\theta) = \left(\sqrt{2}, \frac{\pi}{4}\right)$

b) $(-3,0) \Rightarrow r = \sqrt{(-3)^2 + (0)^2} = 3$

$\cos\theta = -\frac{3}{3} = -1$
$\sin\theta = \frac{0}{3} = 0$
$\theta = \pi$

$(r,\theta) = (3,\pi)$

c) $(\sqrt{3}, -1) \Rightarrow r = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2$.

$\cos\theta = \frac{\sqrt{3}}{2}$
$\sin\theta = \frac{-1}{2}$
in the 4^{th} quadrant
$\theta = \frac{11\pi}{6}$

$(r,\theta) = \left(2, \frac{11\pi}{6}\right)$

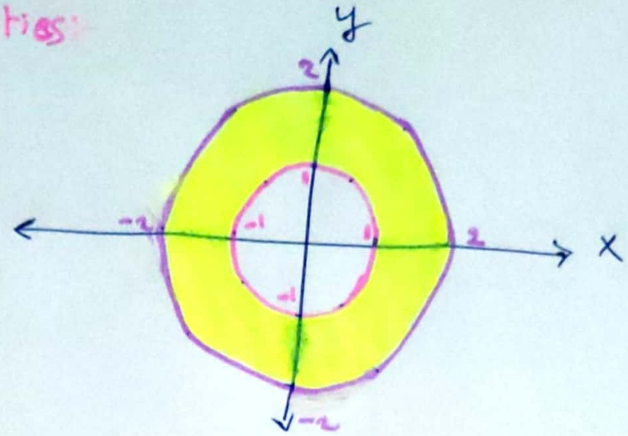
d) $(-3,4) \Rightarrow r = \sqrt{(-3)^2 + (4)^2} = 5$.

$\cos\theta = -\frac{3}{5}$
$\sin\theta = \frac{4}{5}$
in the 2^{nd} quadrant
$\theta = \pi - \tan^{-1}\left(\frac{4}{3}\right)$

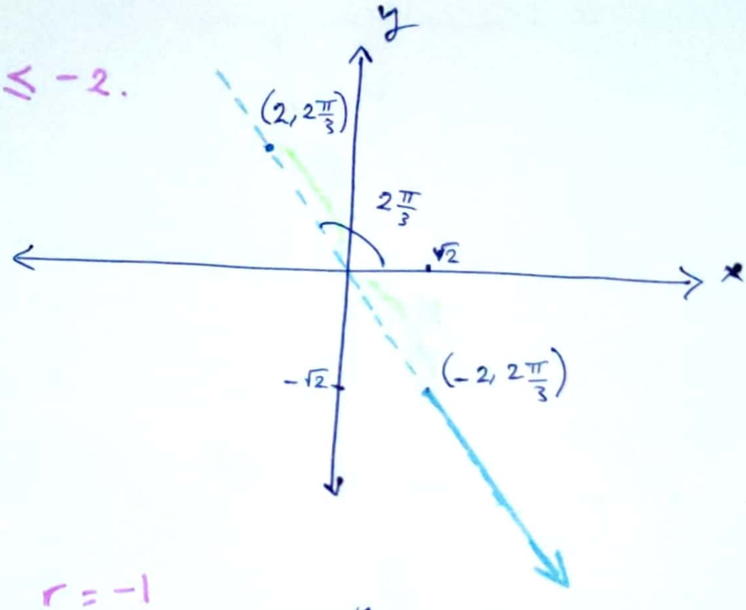
$(r,\theta) = \left(5, \pi - \tan^{-1}\left(\frac{4}{3}\right)\right)$

Graph the sets of points whose polar coordinates satisfy the equations & inequalities:

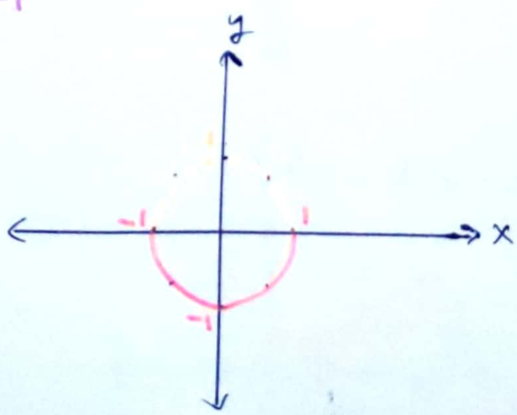
14) $1 \le r \le 2$.



16) $\theta = \frac{2\pi}{3}$, $r \le -2$.



22) $0 \le \theta \le \pi$, $r = -1$



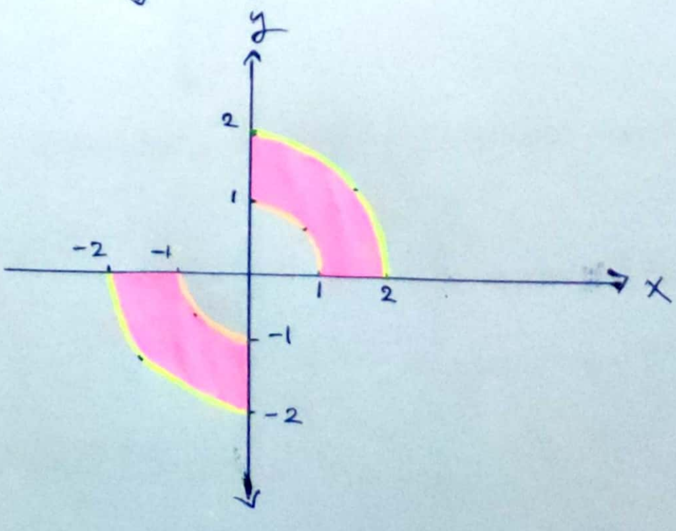
26) $0 \le \theta \le \frac{\pi}{2}$, $1 \le |r| \le 2$.

which mean

$1 \le r \le 2$.

or $-1 \le -r \le -2$

$\Rightarrow -2 \le r \le -1$.

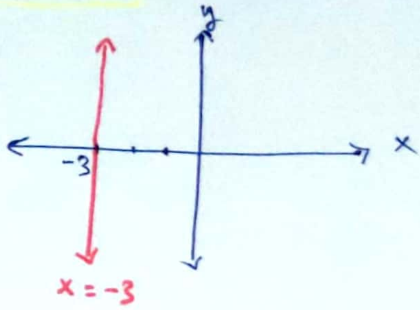


Replace the polar equations with equivalent Cartesian equations. Then describe the graph.

32 $r = -3\sec\theta$.

$r = -3 \cdot \frac{1}{\cos\theta} \Rightarrow r\cos\theta = -3$.

$\boxed{x = -3}$ (vertical line through $(-3,0)$).



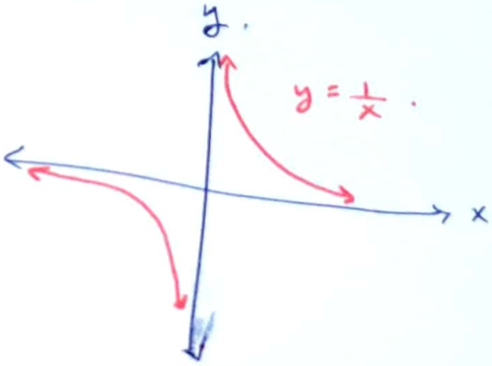
38 $r^2\sin(2\theta) = 2$.

$r^2 \cdot 2\sin\theta\cos\theta = 2$.

$r\sin\theta \cdot r\cos\theta = 1$

$y \quad x \quad = 1$

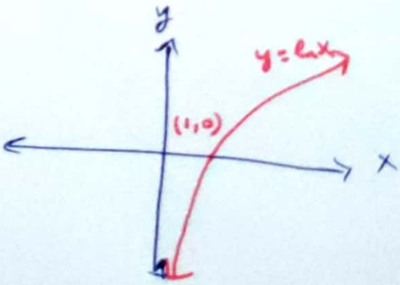
$\boxed{y = \frac{1}{x}}$



42 $r\sin\theta = \ln r + \ln(\cos\theta)$ $\quad$ $(\ln a + \ln b = \ln(ab))$

$r\sin\theta = \ln(r\cos\theta)$.

$\boxed{y = \ln x}$.



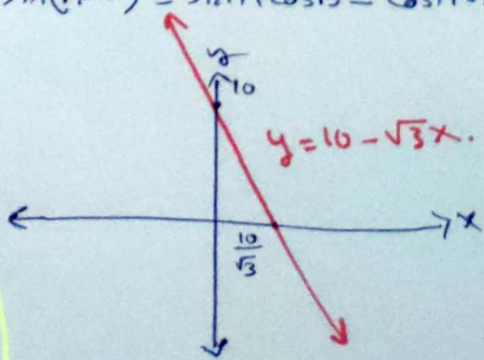
52 $r\sin\left(\frac{2\pi}{3} - \theta\right) = 5$. $\quad$ ☆Remember: $\sin(A-B) = \sin A\cos B - \cos A\sin B$.

$r\left(\sin\left(\frac{2\pi}{3}\right)\cos\theta - \cos\left(\frac{2\pi}{3}\right)\sin\theta\right) = 5$.

$r\left(\frac{\sqrt{3}}{2}\cos\theta - -\frac{1}{2}\sin\theta\right) = 5$

$\frac{\sqrt{3}}{2}r\cos\theta + \frac{1}{2}r\sin\theta = 5$.

$\frac{\sqrt{3}}{2}x + \frac{1}{2}y = 5 \Rightarrow \boxed{y = 10 - \sqrt{3}\,x}$



62. $x^2 + xy + y^2 = 1$. Find Polar equation.

$$x^2 + y^2 + xy = 1.$$

$$r^2 + (r\cos\theta)(r\sin\theta) = 1.$$

$$r^2 + r^2\cos\theta\sin\theta = 1.$$

$$r^2\left(1 + \frac{1}{2}\sin 2\theta\right) = 1.$$

11.4 : Graphing in Polar Coordinates.

Identify the symmetric of the curves. Then sketch the curve

1 $r = 1 + \cos\theta$.

a) Check the symmetry about the x-axis.

$(r, -\theta) \Rightarrow r = 1 + \cos(-\theta)$

$= 1 + \cos\theta$

$= r$

$\Rightarrow (r, \theta)$

Symmetric about the x-axis.

b) Check the symmetry about the y-axis.

$(-r, -\theta) \Rightarrow \quad -r \stackrel{?}{=} 1 + \cos(-\theta)$

$-r \neq 1 + \cos(\theta)$

Cannot tell.

$(r, \pi - \theta) \Rightarrow \quad r \stackrel{?}{=} 1 + \cos(\pi - \theta)$

$\stackrel{?}{=} 1 + \cos(\pi)\cos(\theta) + \sin(\pi)\sin\theta.$

$r \neq 1 - \cos(\theta)$

not symmetric about y-axis.

Therefore not symmetric about the origin. (You can check)

c) Check the symmetry about the origin.

$(-r, \theta) \Rightarrow -r \stackrel{?}{=} 1 + \cos\theta$

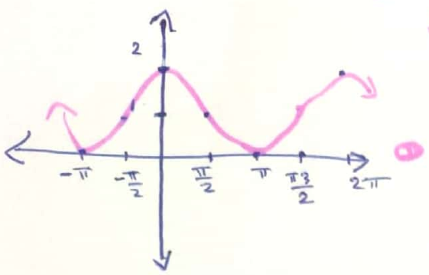
$(r, \theta + \pi) \Rightarrow r \stackrel{?}{=} 1 + \cos(\theta + \pi)$

$\stackrel{?}{=} 1 + \cos(\theta)\cos(\pi) - \sin(\theta)\sin(\pi)$

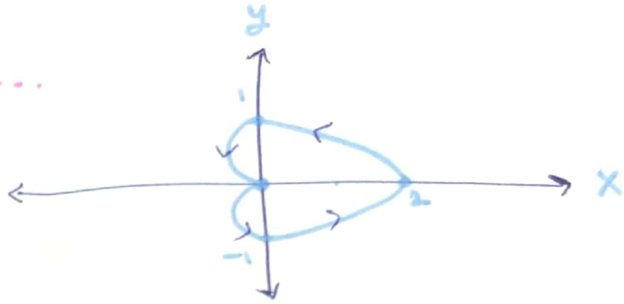
$r \neq 1 - \cos(\theta)$.

$r = 1 + \cos\theta = 0$

$\Rightarrow \theta = \pi, -\pi, 3\pi, \ldots$



$\Rightarrow$



$(r, \theta) \Rightarrow (x, y)$

s.t. $x = r\cos\theta$

$y = r\sin\theta$.

$(2, 0) \Rightarrow (2, 0)$

$(1, \frac{\pi}{2}) \Rightarrow (0, 1)$

$(0, \pi) \Rightarrow (0, 0)$

$(1, \frac{3\pi}{2}) \Rightarrow (0, -1)$

From the graph: No symmetry about y-axis or origin.

$\boxed{6}$ $r = 1 + 2\sin\theta$.

(a) x-axis.

$(r, -\theta) \Rightarrow r \overset{?}{=} 1 + 2\sin(-\theta)$

$\overset{?}{=} 1 - 2\sin\theta$.

$\neq r$

$(-r, \pi - \theta) \Rightarrow -r \overset{?}{=} 1 + 2(\sin(\pi - \theta))$.

$-r \overset{?}{=} 1 + 2\left[\sin(\pi)\cos(\theta) - \cos(\pi)\sin(\theta)\right]$

$\overset{?}{=} 1 + 2\left[0 - -\sin\theta\right]$.

$-r \neq 1 + 2\sin\theta$

not symmetric about x-axis.

b) y-axis.

$(-r, -\theta) \Rightarrow -r \stackrel{?}{=} 1 + 2\sin(-\theta)$.

$-r \neq 1 - 2\sin\theta$

$(r, -\theta) \Rightarrow r \stackrel{?}{=} 1 + 2\sin(\pi - \theta)$.

$r \stackrel{?}{=} 1 + 2[\sin(\pi)\cos(\theta) - \cos(\pi)\sin(\theta)]$.

$r \stackrel{?}{=} 1 + 2[0 + \sin\theta]$

$r = 1 + 2\sin\theta$.

Symmetric about the y-axis.

from a & b $\Rightarrow$ not symmetric about the origin.

c) about the origin.

$(-r, \theta) \Rightarrow -r \neq 1 + 2\sin\theta$.

$(r, \pi + \theta) \Rightarrow r \stackrel{?}{=} 1 + 2\sin(\pi + \theta)$

$\stackrel{?}{=} 1 + 2[\sin(\pi)\cos(\theta) + \cos(\pi)\sin(\theta)]$.

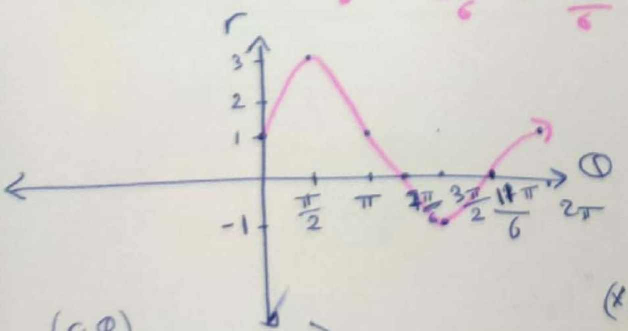
$\neq 1 - 2\sin\theta$.

not symmetric about the origin.

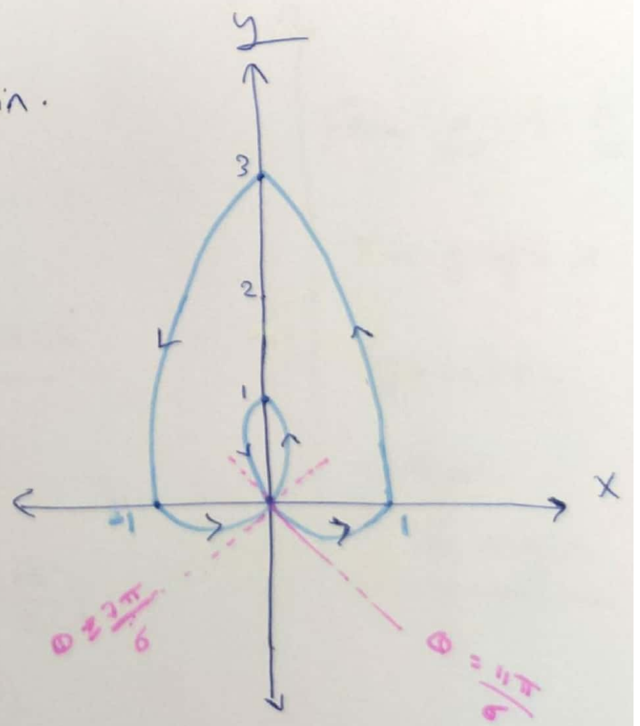
$\Rightarrow r = 1 + 2\sin\theta = 0$

$\sin\theta = -\frac{1}{2}$

$\theta = -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \ldots$

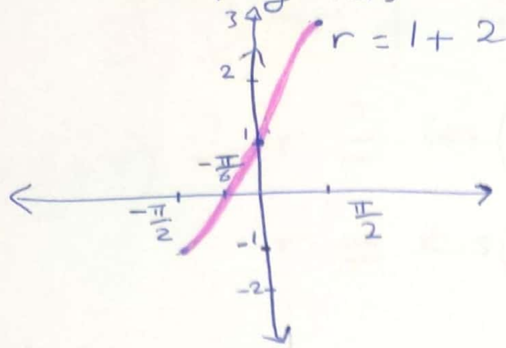


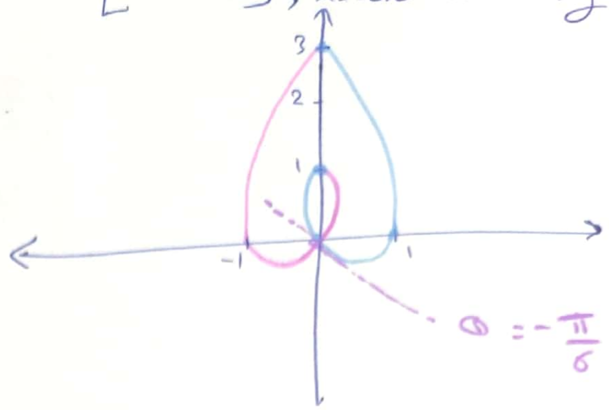
$\Rightarrow$



(r, θ)	(x, y)
$(1, 0)$	$(1, 0)$
$(3, \frac{\pi}{2})$	$(0, 3)$
$(1, \pi)$	$(-1, 0)$
$(0, \frac{7\pi}{6})$	$(0, 0)$
$(-1, \frac{3\pi}{2})$	$(0, 1)$

Note You can have a graph on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then use symmetry about y-axis.





(r, θ)	$\Rightarrow$	(x, y)
$(-1, -\frac{\pi}{2})$	$\Rightarrow$	$(0, 1)$
$(0, -\frac{\pi}{6})$	$\Rightarrow$	$(0, 0)$
$(1, 0)$	$\Rightarrow$	$(1, 0)$
$(3, \frac{\pi}{2})$	$\Rightarrow$	$(0, 3)$

From the graph: No symmetry about x-axis or origin.

---

8 $r = \cos\left(\frac{\theta}{2}\right)$

a) x-axis:

$(r, -\frac{\theta}{2}) \Rightarrow r \stackrel{?}{=} \cos\left(-\frac{\theta}{2}\right)$

$r = \cos\left(-\frac{\theta}{2}\right)$.

The graph is symmetric about x-axis.

b) y-axis:

$(-r, -\frac{\theta}{2}) \Rightarrow -r \stackrel{?}{=} \cos\left(\frac{-\theta}{2}\right)$

$-r \neq \cos\left(\frac{\theta}{2}\right)$

$(r, \pi - \frac{\theta}{2}) \Rightarrow r \stackrel{?}{=} \cos\left(\frac{2\pi - \theta}{2}\right) = \cos\left(\frac{\theta}{2}\right) = r$

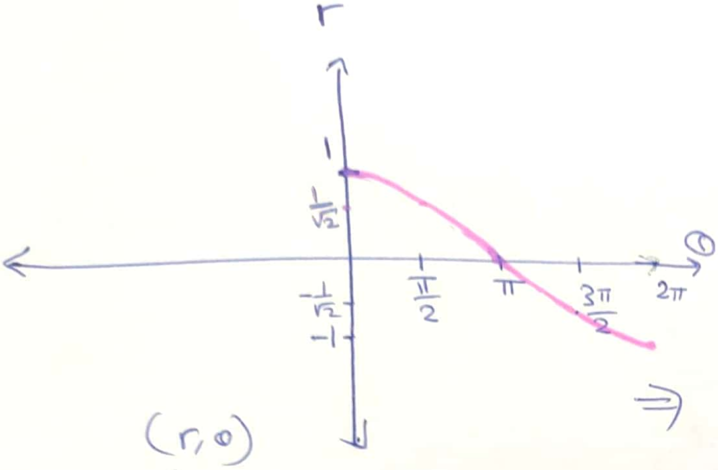
The graph is symmetric about y-axis.

From a & b

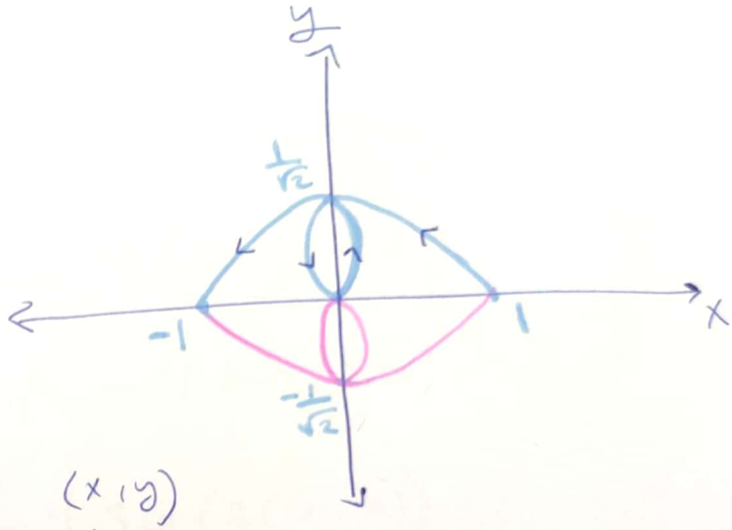
The graph is symmetric about the origin.

$r = \cos\left(\frac{\theta}{2}\right) = 0.$

$\frac{\theta}{2} = \mp\frac{\pi}{2}, \mp\frac{3\pi}{2}, \ldots \Rightarrow \theta = \mp\pi, \mp 3\pi, \ldots$



$\Rightarrow$



(r, θ)	(x, y)
$(1, 0)$	$(1, 0)$
$\left(\frac{1}{\sqrt{2}}, \frac{\pi}{2}\right)$	$\left(0, \frac{1}{\sqrt{2}}\right)$
$(0, \pi)$	$(0, 0)$
$\left(-\frac{1}{\sqrt{2}}, \frac{3\pi}{2}\right)$	$\left(0, \frac{1}{\sqrt{2}}\right)$
$(-1, 2\pi)$	$(-1, 0)$

Note that:

From The graph $\Rightarrow$ The curve has all symmetries.

14 What symmetric does curve have?

$$r^2 = 4\sin(2\theta).$$

x-axis :- $(r, -\theta) \Rightarrow r^2 \overset{?}{=} 4(\sin(-2\theta))$

$$r^2 \neq -4\sin(2\theta)$$

$(-r, \pi - \theta) \Rightarrow (-r)^2 \overset{?}{=} 4\sin(2(\pi-\theta))$

$$r^2 \overset{?}{=} 4\sin(2\pi - 2\theta).$$

$$\overset{?}{=} 4[\sin(2\pi)\cos(2\theta) - \cos(2\pi)\sin(2\theta)]$$

$$\overset{?}{=} 4[0 - \sin(2\theta)].$$

$$r^2 \neq -4\sin(2\theta).$$

y-axis :- $(-r, -\theta) \Rightarrow (-r)^2 \overset{?}{=} 4(\sin(-2\theta))$

$$r^2 \neq -4\sin(2\theta).$$

$(r, \pi - \theta) \Rightarrow r^2 \overset{?}{=} 4\sin(2(\pi - \theta))$

$$\overset{?}{=} 4\sin(2\pi - 2\theta)$$

$$\overset{?}{=} 4[\sin(2\pi)\cos(2\theta) - \cos(2\pi)\sin(2\theta)]$$

$$r^2 \neq -\sin(2\theta).$$

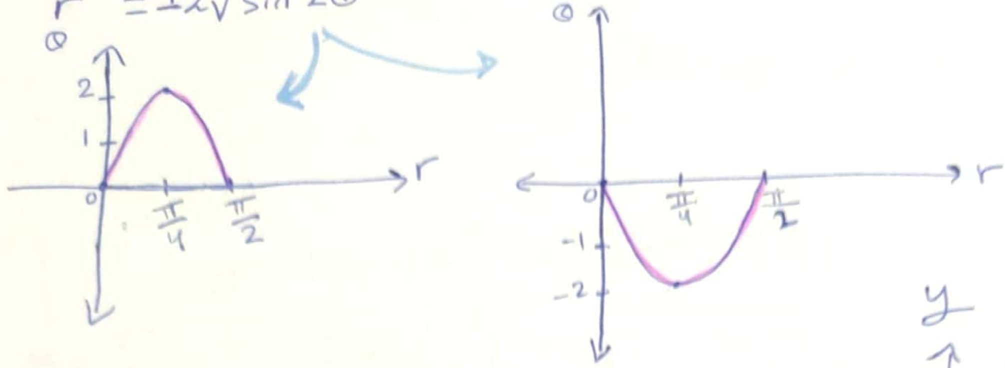
origin :- $(-r, \theta) \Rightarrow (-r)^2 \overset{?}{=} 4\sin(2\theta).$

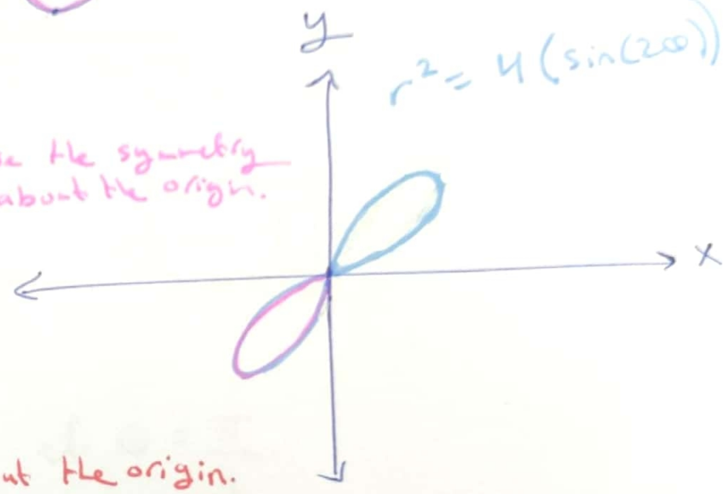
$$r^2 = 4\sin(2\theta)$$

The graph is symmetric about the origin.

$$r^2 = 4\sin(2\theta).$$

$r = \pm 2\sqrt{\sin 2\theta}$, $\sin(2\theta) \geq 0$, $\sin(2\theta) = 0$
$2\theta = 0, \pi, 2\pi \Rightarrow \theta = 0, \frac{\pi}{2}$





(r, θ)	$\Rightarrow$	(x, y)
$(0, 0)$	$\Rightarrow$	$(0, 0)$
$(2, \frac{\pi}{4})$	$\Rightarrow$	$(\sqrt{2}, \sqrt{2})$
$(0, \frac{\pi}{2})$	$\Rightarrow$	$(0, 0)$

use the symmetry about the origin.

From the graph:
The graph only symmetric about the origin.

19 Find the slope of the curve.

$r = \sin(2\theta)$, at $\theta = \mp \frac{\pi}{4}$, $\mp \frac{3\pi}{4}$.

$\theta = -\frac{\pi}{4}$, $r = -1$ $(-1, -\frac{\pi}{4})$, $\theta = \frac{\pi}{4}$, $r = 1$ $(1, \frac{\pi}{4})$, $\theta = -\frac{3\pi}{4}$, $r = 1$, $(1, -\frac{3\pi}{4})$, $\theta = \frac{3\pi}{4}$, $r = -1$ $(-1, \frac{3\pi}{4})$

remember :>
$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}.$$

$$r = f(\theta) = \sin(2\theta).$$

$$r' = f'(\theta) = 2\cos(2\theta).$$

$$y' = \frac{2\cos(2\theta)\sin(\theta) + \sin(2\theta)\cos(\theta)}{2\cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta)}$$

$$\boxed{y'\Big|_{\theta = -\frac{\pi}{4}}} = \frac{2\cos(-2\frac{\pi}{4})\sin(-\frac{\pi}{4}) + \sin(-2\frac{\pi}{4})\cos(-\frac{\pi}{4})}{2\cos(-2\frac{\pi}{4})\cos(-\frac{\pi}{4}) - \sin(-\frac{2\pi}{4})\sin(-\frac{\pi}{4})}$$

$$= \frac{-2\cos(\frac{\pi}{2})\sin(\frac{\pi}{4}) - \sin(\frac{\pi}{2})\cos(\frac{\pi}{4})}{2\cos(\frac{\pi}{2})\cos(\frac{\pi}{4}) - \sin(\frac{\pi}{2})\sin(\frac{\pi}{4})}$$

$$y'\Big|_{\theta=-\frac{\pi}{4}} = \frac{0-(1)\left(\frac{1}{\sqrt{2}}\right)}{0-1\left(\frac{1}{\sqrt{2}}\right)} = \boxed{1}$$

= The slop at $\theta = -\frac{\pi}{4}$.

$$\boxed{y'\Big|_{\theta=\frac{\pi}{4}}} = \frac{2\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{4}\right)}{2\cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{4}\right)-\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{4}\right)}$$

$$= \frac{0+(1)\frac{1}{\sqrt{2}}}{0-(1)\frac{1}{\sqrt{2}}} = \boxed{-1}$$

= the slope at $\theta = \frac{\pi}{4}$.

$$\boxed{y'\Big|_{\theta=-\frac{3\pi}{4}}} = \frac{2\cos\left(-\frac{3\pi}{2}\right)\sin\left(-\frac{3\pi}{4}\right)+\sin\left(-\frac{3\pi}{2}\right)\cos\left(-\frac{3\pi}{4}\right)}{2\cos\left(-\frac{3\pi}{2}\right)\cos\left(-\frac{3\pi}{4}\right)-\sin\left(-\frac{3\pi}{2}\right)\sin\left(-\frac{3\pi}{4}\right)}$$

$$= \frac{0+-(-1)\left(-\frac{1}{\sqrt{2}}\right)}{0-(-(-1))\left(-\frac{1}{\sqrt{2}}\right)}$$

$$= \frac{-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \boxed{-1}$$

= The slope at $\theta = \frac{-3\pi}{4}$.

$$\left. y' \right|_{\theta = \frac{3\pi}{4}} = \frac{2\cos\left(\frac{3\pi}{2}\right)\sin\left(\frac{3\pi}{4}\right) + \sin\left(\frac{3\pi}{2}\right)\cos\left(\frac{3\pi}{4}\right)}{2\cos\left(\frac{3\pi}{2}\right)\cos\left(\frac{3\pi}{4}\right) - \sin\left(\frac{3\pi}{2}\right)\sin\left(\frac{3\pi}{4}\right)}$$

$$= \frac{0 + (-1)\left(-\frac{1}{\sqrt{2}}\right)}{0 - (-1)\left(\frac{1}{\sqrt{2}}\right)}$$

$$= \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$$

$$= \boxed{1}$$

= The slope at $\theta = \frac{3\pi}{4}$.

Check the symmetry:

x-axis $\Rightarrow (r, -\theta) \Rightarrow r \overset{?}{=} \sin(-2\theta)$

$r \neq 2\sin(2\theta)$

$(-r, \pi - \theta) \Rightarrow -r \overset{?}{=} \sin(2\pi - 2\theta)$

$\overset{?}{=} \sin(2\pi)\cos(2\theta) - \cos(2\pi)\sin(2\theta)$.

$\overset{?}{=} 0 - (1)(\sin(2\theta))$

$-r = -\sin(2\theta)$.

∴ The curve is symmetric about x-axis.

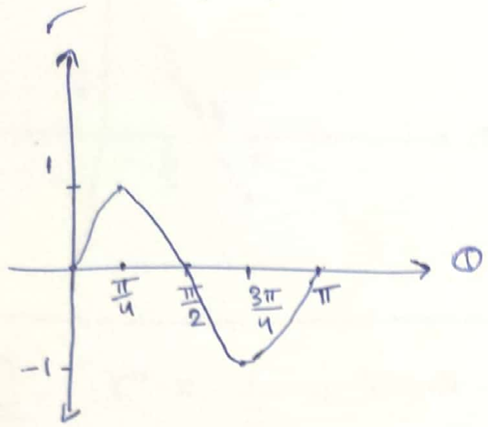
y-axis :- $(-r, -\theta) \Rightarrow -r \overset{?}{=} \sin(-2\theta)$.

$-r = -\sin(2\theta)$.

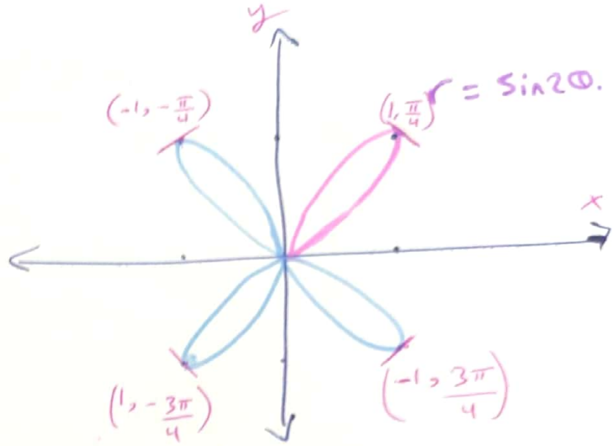
∴ The curve is symmetric about y-axis.

∴ So, the curve is symmetric about the origin.

$r = \sin(2\theta) = 0 \Rightarrow 2\theta = 0, \pi, 2\pi \Rightarrow \theta = 0, \frac{\pi}{2}, \pi.$



$\Rightarrow$



$(r, \theta) \Rightarrow (x, y)$
$(0, 0) \Rightarrow (0, 0)$
$(1, \frac{\pi}{4}) \Rightarrow (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$
$(0, \frac{\pi}{2}) \Rightarrow (0, 0)$

Then use the symmetry

$\boxed{21}$ Graph:

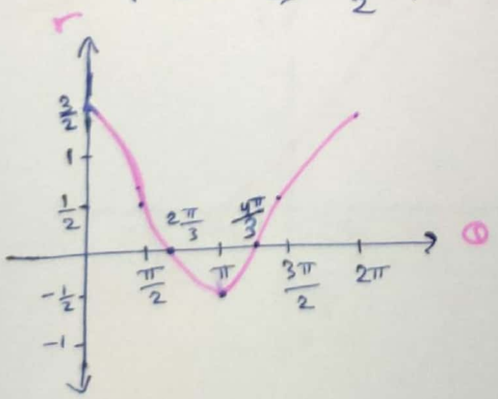
$\boxed{a}$ $r = \frac{1}{2} + \cos\theta.$

The curve is symmetric the x-axis. Check it

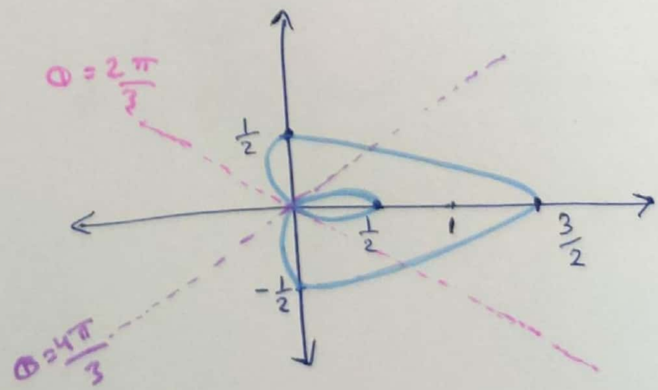
$r = \frac{1}{2} + \cos\theta.$

$r = 0 \Rightarrow \frac{1}{2} + \cos\theta = 0 \Rightarrow \cos\theta = -\frac{1}{2}$

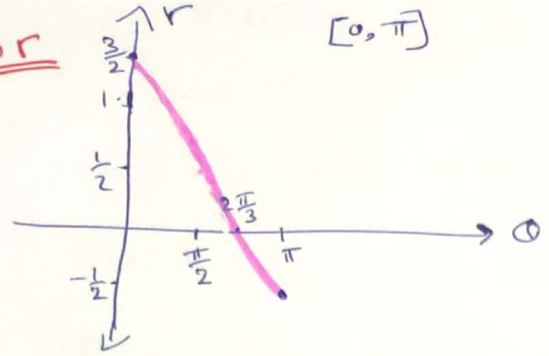
$\Rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}.$



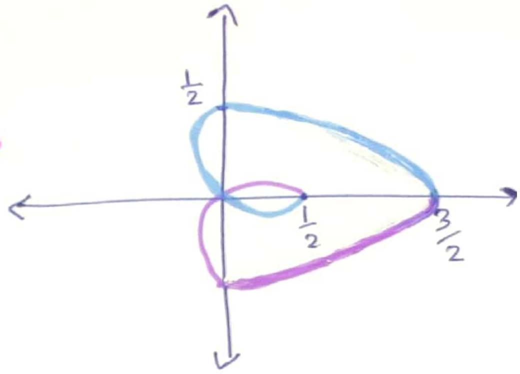
$\Rightarrow$



or



use symmetry about x-axis

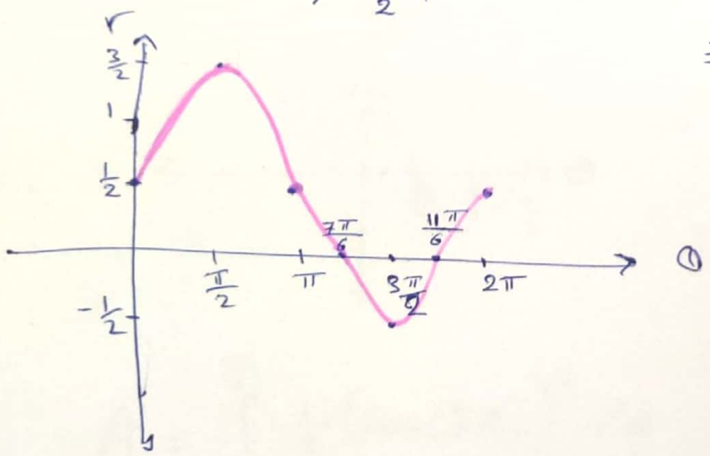


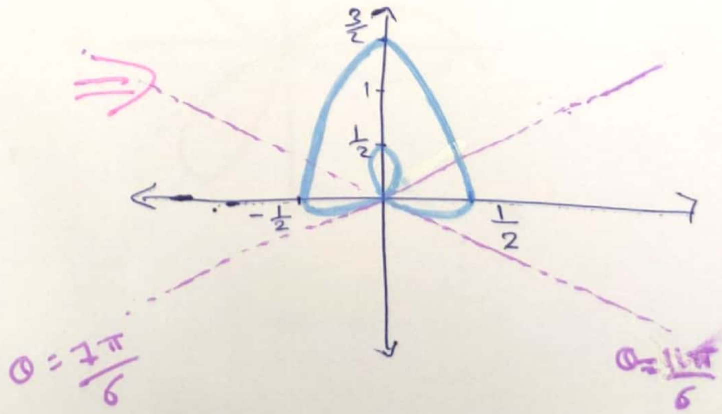
b $\quad r = \frac{1}{2} + \sin\theta$.

The curve is symmetric about y-axis check it

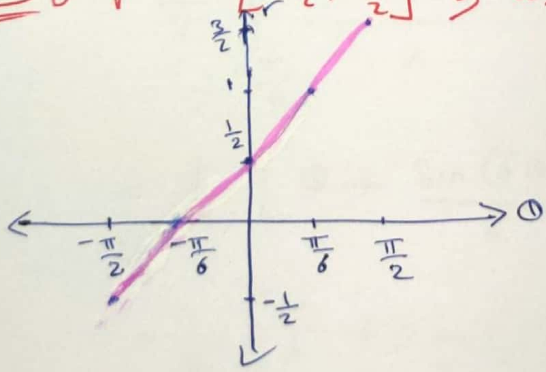
$r = 0 \Rightarrow \frac{1}{2} + \sin\theta = 0 \Rightarrow \sin\theta = -\frac{1}{2}$

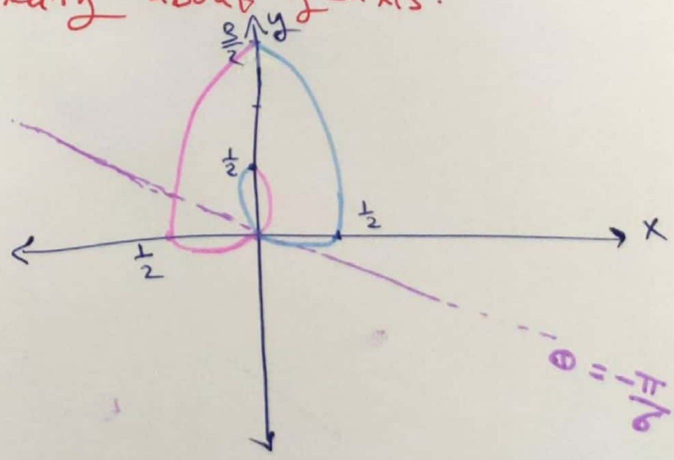
$\Rightarrow \theta = -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$





or graph on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ & use symmetry about y-axis.





11.5 Areas & Lengths in Polar Coordinates.

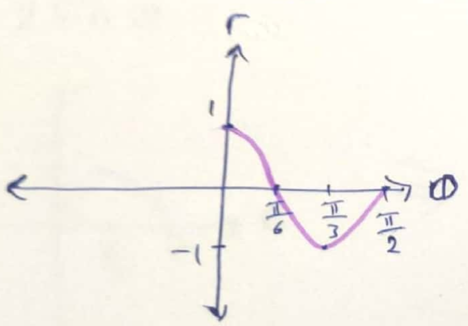
6 Find the area of region inside one leaf of the three-leaved rose $r = \cos(3\theta)$.

$\star$ symmetric about x-axis (check it).

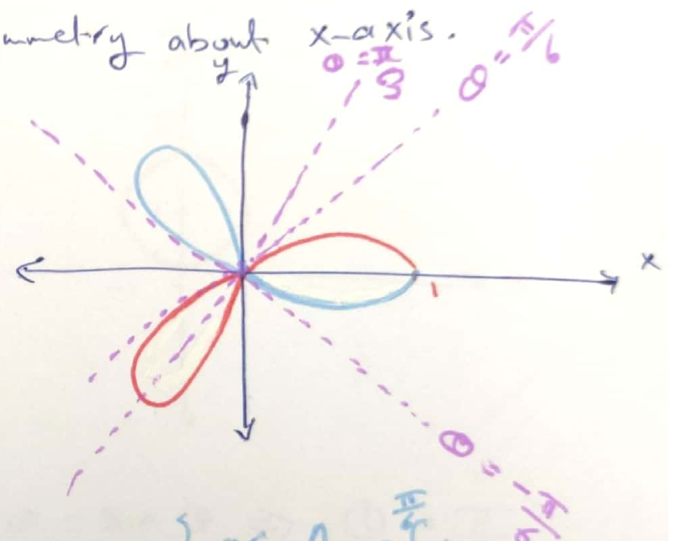
$\cos(3\theta) = 0 \Rightarrow 3\theta = \mp\frac{\pi}{2}, \mp\frac{3\pi}{2}$

$\Rightarrow \theta = \mp\frac{\pi}{6}, \mp\frac{\pi}{2}.$

graph on $[0, \frac{\pi}{2}]$ then use symmetry about x-axis.



$\Longrightarrow$



$$A = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}\left(\cos(3\theta)\right)^2 d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1+\cos(6\theta)}{2}\, d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{\sin(6\theta)}{6}\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$

$$= \frac{1}{4}\left[\frac{\pi}{6} + 0 - \left(-\frac{\pi}{6} + 0\right)\right]$$

$$= \frac{1}{4}\left[\frac{\pi}{6} + \frac{\pi}{6}\right]$$

$$= \boxed{\frac{\pi}{12}}$$

or $A = 2\int_{0}^{\frac{\pi}{6}} \frac{1}{2}\cos^2(3\theta)\, d\theta = \frac{\pi}{12}$

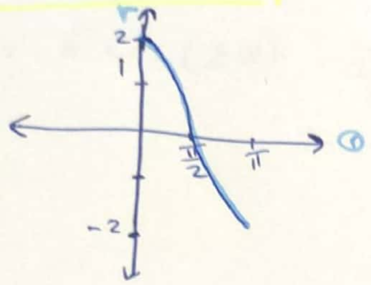
or $A = 2\int_{-\frac{\pi}{6}}^{0} \frac{1}{2}\cos^2(3\theta)\, d\theta$

$= \frac{\pi}{12}$

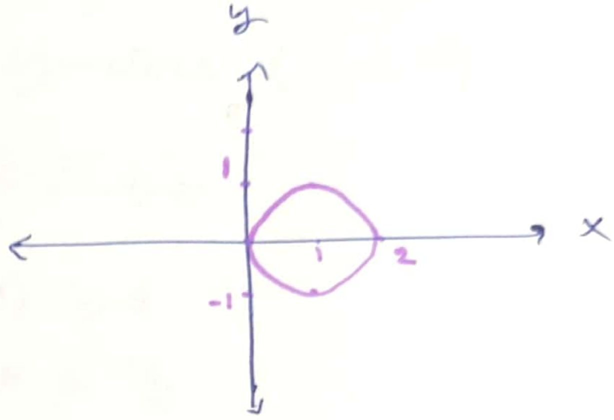
$\boxed{9}$ Find the area of the region shared by the circles

$r = 2\cos\theta$ & $r = 2\sin\theta$.

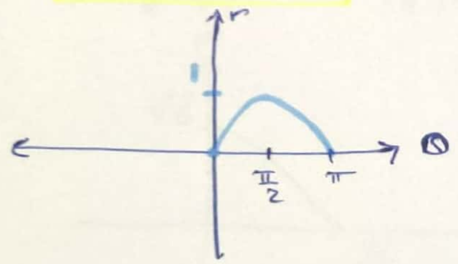
$r = 2\cos\theta$:



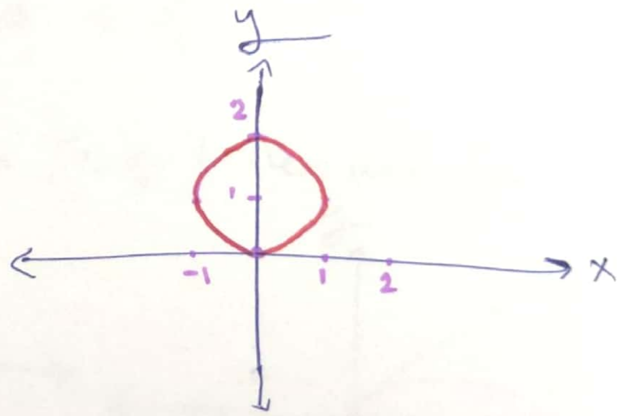
$\Rightarrow$



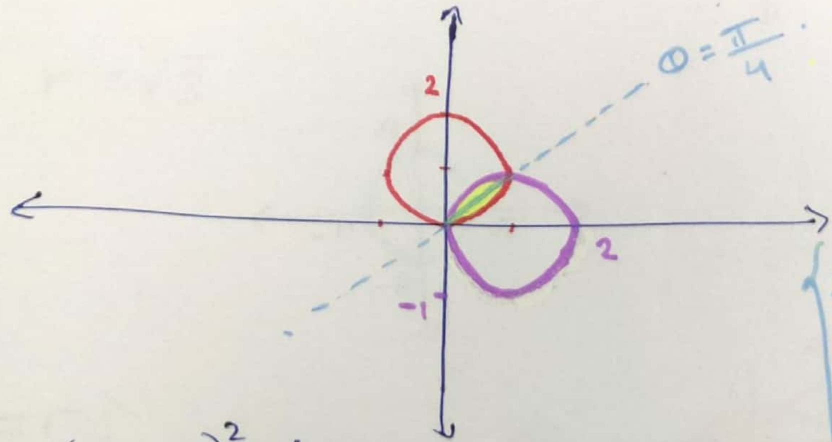
$r = 2\sin\theta$



$\Rightarrow$



$2\cos\theta = 2\sin\theta \Rightarrow \cos\theta = \sin\theta \Rightarrow \boxed{\theta = \frac{\pi}{4}}$



or
$A = 2\int_{\pi/4}^{\pi/2} \frac{1}{2}(2\cos$

$= \frac{\pi}{2} - 1$

$$A = 2\int_0^{\pi/4} \frac{1}{2}(2\sin\theta)^2\, d\theta$$

$$= 4\int_0^{\pi/4} \sin^2\theta\, d\theta = 4\int_0^{\pi/4} \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= 2\left[\theta - \frac{\sin(2\theta)}{2}\Big|_0^{\pi/4}\right] = 2\left[\frac{\pi}{4} - \frac{\sin(\frac{\pi}{2})}{2}\right] = \boxed{\frac{\pi}{2} - 1}$$

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13 Find the area of the region <u>inside</u> the lemniscate $r^2 = 6\cos(2\theta)$ & <u>outside</u> the circle $r = \sqrt{3}$.

$r^2 = 6\cos(2\theta)$ has all symmetries (check it).

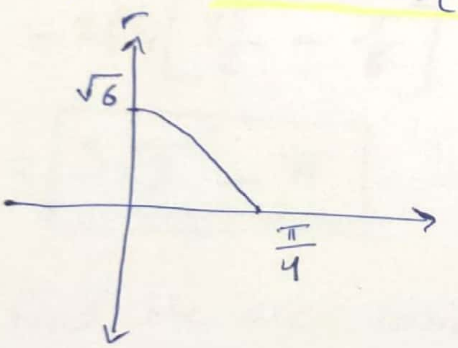
$r^2 = 6\cos(2\theta) \Rightarrow r = \sqrt{6}\sqrt{\cos(2\theta)}$

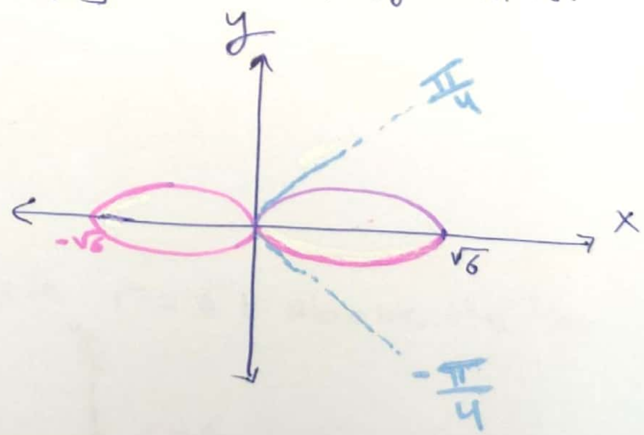
$\cos(2\theta) \geq 0$

$0 \leq 2\theta \leq \frac{\pi}{2}$.

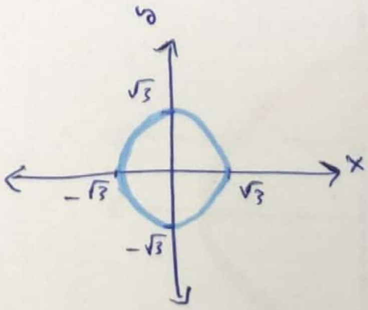
$0 \leq \theta \leq \frac{\pi}{4}$.

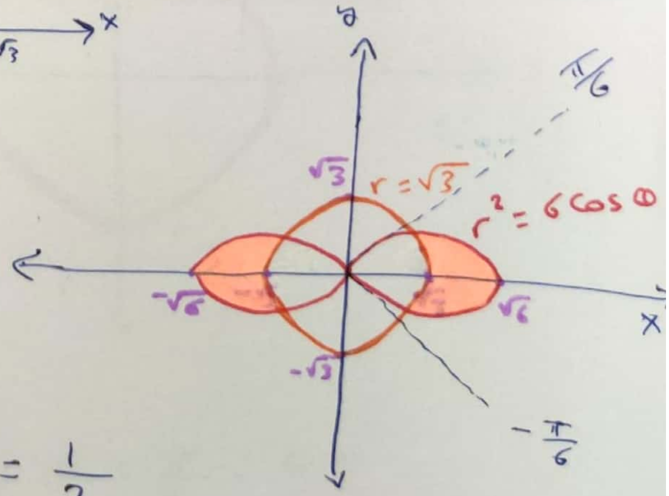
$\Rightarrow$ graph $r = \sqrt{6}\sqrt{\cos(2\theta)}$ on $[0, \frac{\pi}{4}]$ then use symmetries.





$\Rightarrow$ graph $r = \sqrt{3}$





$r = r$

$\sqrt{6}\sqrt{\cos(2\theta)} = \sqrt{3}$

$\Rightarrow r^2 = r^2$

$6\cos(2\theta) = 3 \Rightarrow \cos(2\theta) = \frac{1}{2}$

$\Rightarrow 2\theta = \mp\frac{\pi}{3}$.

$\Rightarrow \theta = \mp\frac{\pi}{6}$

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$$A = 4\int_0^{\frac{\pi}{6}} \left[\frac{1}{2}\left(\sqrt{6}\sqrt{\cos(2\theta)}\right)^2 - \frac{1}{2}(\sqrt{3})^2\right] d\theta.$$

$$= 4\int_0^{\frac{\pi}{6}} \left[\frac{1}{2}\, 6\cos(2\theta) - \frac{3}{2}\right] d\theta$$

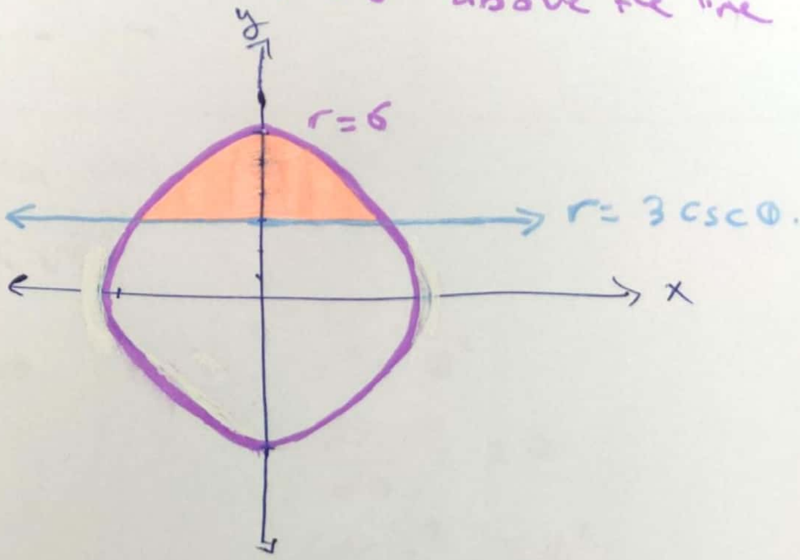
$$= 2\left[\frac{6\sin(2\theta)}{2} - 3\theta \Big|_0^{\frac{\pi}{6}}\right].$$

$$= 2(3)\left[\sin\left(2\cdot\frac{\pi}{6}\right) - \frac{\pi}{6} - (0)\right].$$

$$= 2(3)\left[\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right]$$

$$= \boxed{3\sqrt{3} - \pi}$$

16) Find the area inside the circle $r = 6$ above the line $r = 3\csc\theta$.



$r = 6$

$r = 3\csc\theta$

$r = 3\,\frac{1}{\sin\theta}$

$r\sin\theta = 3$

$y = 3.$

$3\csc\theta = 6$

$\frac{3}{\sin\theta} = 6 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$$A = \int_{\pi/6}^{5\pi/6} \left(\frac{1}{2}(6)^2 - \frac{1}{2}(3\csc\theta)^2 \right) d\theta$$

$$= \int_{\pi/6}^{5\pi/6} \left(18 - \frac{9}{2}\csc^2\theta \right) d\theta$$

$$= \left(18\theta + \frac{9}{2}\cot\theta \right) \Big|_{\pi/6}^{5\pi/6}$$

$$= 18\frac{(5\pi)}{6} + \frac{9}{2}\cot\left(\frac{5\pi}{6}\right) - 18\frac{\pi}{6} - \frac{9}{2}\cot\left(\frac{\pi}{6}\right)$$

$$= 15\pi + \frac{9}{2}(-\sqrt{3}) - 3\pi - \frac{9}{2}\sqrt{3}$$

$$= \boxed{12\pi - 9\sqrt{3}}$$

---

21) Find the length of the curve

$r = \theta^2 \quad , \quad 0 \leq \theta \leq \sqrt{5}.$

$$L = \int_0^{\sqrt{5}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta.$$

$r = \theta^2$

$r^2 = \theta^4$

$\frac{dr}{d\theta} = 2\theta$

$\left(\frac{dr}{d\theta}\right)^2 = 4\theta^2.$

$$= \int_0^{\sqrt{5}} \sqrt{\theta^4 + 4\theta^2} \, d\theta.$$

$$= \int_0^{\sqrt{5}} \sqrt{\theta^2}\sqrt{\theta^2+4} \, d\theta = \int_0^{\sqrt{5}} |\theta| \sqrt{\theta^2+4} \, d\theta.$$

but $|\theta| = \theta$ since $\theta \geq 0$.

So, $\displaystyle\int_0^{\sqrt{5}} \theta\sqrt{\theta^2+4}\, d\theta.$

let $u = \theta^2+4$.

$du = 2\theta\, d\theta$.

$\frac{du}{2} = \theta\, d\theta$

when $\theta = 0 \Rightarrow u = 4$.

$\theta = \sqrt{5} \Rightarrow u = 5+4 = 9$.

$$= \frac{1}{2}\int_4^9 \sqrt{u}\, du$$

$$= \frac{1}{2}\,\frac{2}{3}u^{\frac{3}{2}}\Big|_4^9$$

$$= \frac{1}{3}\left[9^{\frac{3}{2}} - 4^{\frac{3}{2}}\right].$$

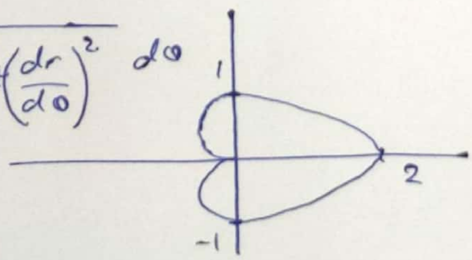
$$= \frac{1}{3}\left[\left(9^{\frac{1}{2}}\right)^3 - \left(4^{\frac{1}{2}}\right)^3\right]$$

$$= \frac{1}{3}\left[27-8\right)$$

$$= \boxed{\frac{19}{3}}$$

23) Find the length of the cardioid $r = 1+\cos\theta$.

$$L = \int_0^{2\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta. = 2\int_0^{\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta$$



$$= 2\int_0^{\pi}\sqrt{1+2\cos\theta+\underbrace{\cos^2\theta+\sin^2\theta}_{1}}\, d\theta.$$

$r = 1+\cos\theta$

$r^2 = 1+2\cos\theta+\cos^2\theta.$

$\frac{dr}{d\theta} = -\sin\theta$

$\left(\frac{dr}{d\theta}\right)^2 = \sin^2\theta.$

$$= 2\int_0^{\pi}\sqrt{2+2\cos\theta}\, d\theta.$$

$$= 2\int_0^{\pi} \sqrt{\frac{4}{2}(1+\cos\theta)}\ d\theta.$$

$$= 2\int_0^{\pi} \sqrt{4}\ \sqrt{\frac{1}{2}+\frac{\cos\theta}{2}}\ d\theta.$$

$$= 4\int_0^{2\pi} \cos\left(\frac{\theta}{2}\right) d\theta.$$

$$= 4\ \frac{\sin\left(\frac{\theta}{2}\right)}{\frac{1}{2}}\Bigg|_0^{\pi}$$

$$= 8\left[\sin(\pi) - \sin 0\right]$$

$$= 8\left[1-0\right]$$

$$= \boxed{8}$$

$\cos(2\theta) = 2\cos^2\theta - 1$

$\cos(\theta) = 2\cos^2\left(\frac{\theta}{2}\right) - 1$

$\frac{\cos\theta + 1}{2} = \cos^2\left(\frac{\theta}{2}\right).$