10.4 Comparison Test

- The comparison test: (D.C.T)

Let $\sum a_n$, $\sum c_n$ and $\sum b_n$ be series with nonnegative terms. Suppose that for some integer N.

$$d_n \leq a_n \leq c_n \quad \text{for all } n > N$$

a) If $\sum c_n$ converges, then $\sum a_n$ also converges

b) if $\sum d_n$ diverges, then $\sum a_n$ also diverges

- The limit comparison test (L.C.T)

Suppose that $a_n > 0$, $b_n > 0$ for $n \geq N$

1) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converges or both diverges

2) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges

3) If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Remember that harmonic series $\sum \frac{1}{n}$ diverges and p-series $\sum \frac{1}{n^p}$ converges if $P > 1$ and diverges for $P \leq 1$

Questions: 8, 15, 18, 27, 28, 32, 40, 43, 52

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$\boxed{8}$ use D.C.T to determine if the following sequence converges or diverges.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{\sqrt{n^2+3}} \quad \text{Compare with} \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

(Note that $\frac{\sqrt{n}+1}{\sqrt{n^2+3}}$ and $\frac{1}{\sqrt{n}}$ nonnegative for $n \geq 1$)

(also note that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series, $P = \frac{1}{2}$ which is divergent series)

for $n \geq 1$, $\sqrt{n} \geq 1$, $2\sqrt{n} \geq 2$

$2\sqrt{n} + 1 \geq 3$ multiply both sides by n

$2n\sqrt{n} + n \geq 3n \geq 3$ add n^2 for both sides

$n^2 + 2n\sqrt{n} + n \geq n^2 + 3$

$n(n + 2\sqrt{n} + 1) \geq n^2 + 3$

$$\frac{n(\sqrt{n}+1)^2}{n^2+3} \geq 1$$

$$\frac{(\sqrt{n}+1)^2}{n^2+3} \geq \frac{1}{n} \quad \text{take the square root of both sides}$$

$\frac{\sqrt{n}+1}{\sqrt{n^2+3}} \geq \frac{1}{\sqrt{n}}$ so $\frac{\sqrt{n}+1}{\sqrt{n^2+3}}$ diverges by L.C.T

$\boxed{15}$ use L.C.T. to determine if the following series Converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{\ln n} \quad , \text{ compare with } \sum_{n=2}^{\infty} \frac{1}{n}$$

Note that $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges (harmonic series)

$$\lim_{n\to\infty} \frac{\frac{1}{\ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\ln n} = \lim_{n\to\infty} \frac{1}{\frac{1}{n}} = \lim_{n\to\infty} n = \infty$$

So by L.C.T $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges

$\boxed{18}$ use any method to determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}} \quad , \text{ use L.C.T with } \sum_{n=1}^{\infty} \frac{1}{n}$$

(Note that $\frac{3}{n+\sqrt{n}}$, $\frac{1}{n}$ are positive for $n \geq 1$)

$$\lim_{n\to\infty} \frac{\frac{3}{n+\sqrt{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{3n}{n+\sqrt{n}} = \lim_{n\to\infty} \frac{3}{1+\frac{1}{2\sqrt{n}}} = 3$$

So both Converge or both diverge.

but $\sum \frac{1}{n}$ diverges (harmonic series)

So by L.C.T $\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$ diverges

$\boxed{27}$ $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$ (use any method)

Compare with $\sum_{n=3}^{\infty} \frac{1}{n}$

(Note that $\frac{1}{n}$, $\frac{1}{\ln(\ln n)}$ are positive for $n \geq 3$)

$n > \ln n$ (take ln for both sides)

$\ln n > \ln(\ln n)$

so $n > \ln n > \ln(\ln n)$

$\therefore \frac{1}{n} < \frac{1}{\ln(\ln n)}$

so $\sum_{n=3}^{\infty} \frac{1}{n} < \sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$

but $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges (harmonic series)

so by D.C.T $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$ diverges.

$\boxed{28}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$

use L.C.T, compare with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

Note : $\dfrac{(\ln n)^2}{n^3}$, $\dfrac{1}{n^2}$ are positive for $n \geq 1$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (P-series with $P > 1$)

$$\lim_{n\to\infty} \frac{\frac{(\ln n)^2}{n^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{(\ln n)^2}{n^{3}} \cdot \cancel{n^2} = \lim_{n\to\infty} \frac{(\ln n)^2}{n} = \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{2(\ln n)(\frac{1}{n})}{1} = \lim_{n\to\infty} \frac{2\ln n}{n} \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{2(\frac{1}{n})}{1} = 0$$

so by L.C.T $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$ converges.

$\boxed{32}$ $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n+1)}{n+1}$

use L.C.T compare with $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n+1}$

$$\lim_{n\to\infty} \frac{\frac{\ln(n+1)}{n+1}}{\frac{1}{n+1}} = \lim_{n\to\infty} \ln(n+1) = \infty$$

note that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n+1}$ diverges by the integral test

$$\left(\int_2^{\infty} \frac{dx}{x+1} = \lim_{b\to\infty} \left[\ln|x+1| \Big|_2^b\right] = \lim_{b\to\infty} \left[\ln|b+1| - \ln|3| \right] = \infty \right)$$

so by L.C.T $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n+1)}{n+1}$ diverges

Note that you can use the integral test to show that $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n+1)}{n+1}$ diverges.

$\boxed{40}$ $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 3^n}{3^n + 4^n}$

Compare with $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n}$

Note that $\dfrac{2^n+3^n}{3^n+4^n}$ and $\dfrac{2^n+3^n}{4^n}$ are positive for $n \geq 1$

Note that $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+3^n}{4^n} = \sum_{n=1}^{\infty} \left(\frac{2}{4}\right)^n + \left(\frac{3}{4}\right)^n$

$$= \sum_{n=1}^{\infty} \left(\frac{2}{4}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n = \frac{\frac{2}{4}}{1-\frac{2}{4}} + \frac{\frac{3}{4}}{1-\frac{3}{4}}$$

$$= \frac{\frac{2}{4}}{\frac{2}{4}} + \frac{\frac{3}{4}}{\frac{1}{4}} = 1 + 3 = 4$$

So $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+3^n}{4^n}$

by D.C.T since $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+4^n} \leq \sum_{n=1}^{\infty} \frac{2^n+3^n}{4^n}$

~~So~~ $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+4^n}$ converges.

We can use L.C.T $\left(\lim_{n\to\infty} \dfrac{\frac{2^n+3^n}{3^n+4^n}}{\frac{2^n+3^n}{4^n}} = 1 \right)$

We can use L.C.T $\left(\lim_{n\to\infty} \dfrac{\frac{2^n+3^n}{3^n+4^n}}{\frac{3^n}{4^n}} = 1 \right)$

$\boxed{43}$ $\sum_{n=2}^{\infty} \frac{1}{n!}$, Compare with $\sum_{n=2}^{\infty} \frac{1}{2^n}$

$$\sum_{n=2}^{\infty} \frac{1}{n!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$$

$$\leq \underbrace{\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots}_{\sum_{n=1}^{\infty} \frac{1}{2^n}}$$

Note that $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with $|r| = \left|\frac{1}{2}\right| < 1$, So it converges.

by D.C.T $\sum_{n=2}^{\infty} \frac{1}{n!}$ converges.

also we can compare with $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$

for $n \geq 2$, $(n-2)! \geq 1$

so $n(n-1)(n-2)! \geq n(n-1)$

$$n! \geq n(n-1)$$

$$\frac{1}{n!} \leq \frac{1}{n(n-1)}$$

$$\sum_{n=2}^{\infty} \frac{1}{n!} \leq \sum_{n=2}^{\infty} \frac{1}{n(n-1)}$$

Note that $\sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \sum_{n=2}^{\infty} \left[\frac{1}{n-1} - \frac{1}{n}\right]$

which converges by n^{th} partial sum.

$\boxed{52}$ $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$, compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$

($\frac{\sqrt[n]{n}}{n^2}$, $\frac{1}{n^2}$ are positive for $n \geq 1$)

$$\lim_{n\to\infty} \frac{\frac{\sqrt[n]{n}}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{\cancel{n^2}} \cdot \cancel{n^2}$$

$$= \lim_{n\to\infty} \sqrt[n]{n} = 1$$