10.5 Absolute Convergence ; The Ratio and Root Tests

* $\sum a_n$ Converges absolutely if $\sum |a_n|$ Converges.

* If $\sum |a_n|$ Converges, then $\sum a_n$ converges.

- The Ratio Test:

Let $\sum a_n$ be any series such that

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho$$

a) If $\rho < 1$, then the series converges absolutely

b) if $\rho > 0$ or ρ is infinite, then $\sum a_n$ diverges.

c) if $\rho = 1$, then the test inconclusive.

The Ratio test is ~~[illegible]~~ effective when the terms contain factorials of expressions involving n or expressions raised to a power involving n.

- The Root Test.

Let $\sum a_n$ be any series and suppose that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho$ then

a) $\sum a_n$ Converges absolutely if $\rho < 1$

b) $\sum a_n$ diverges if $\rho > 1$ or ρ is infinite.

c) the test is inconclusive if $\rho = 1$

Questions: 7, 12, 16, 20, 28, 38, 43, 46, 60

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$\boxed{7}$ use the Ratio Test to determine if the following series converges absolutely or diverges.

$$\sum_{n=1}^{\infty} (-1)^{n} \frac{n^2 (n+2)!}{n! \; 3^{2n}}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\dfrac{(n+1)^2 (n+1+2)!}{(n+1)! \; 3^{2(n+1)}}}{\dfrac{n^2 (n+2)!}{n! \; 3^{2n}}} \right|$$

$$= \left| \frac{(n+1)^2 (n+3)!}{(n+1)! \; 3^{2n+2}} \cdot \frac{n! \; 3^{2n}}{n^2 (n+2)!} \right|$$

$$= \left| \frac{(n+1)^2 \cdot (n+3) \cdot \cancel{(n+2)!}}{(n+1) \cancel{n!} \; \cancel{3^{2n}} \cdot 3^2} \cdot \frac{\cancel{n!} \; \cancel{3^{2n}}}{n^2 \cancel{(n+2)!}} \right|$$

$$= \left| \frac{(n+1)^2 (n+3)}{3^2 (n+1) n^2} \right| = \left| \frac{n^3 + 5n^2 + 7n + 3}{9n^3 + 9n^2} \right|$$

$$\lim_{n \to \infty} \left| \frac{n^3 + 5n^2 + 7n + 3}{9n^3 + 9n^2} \right| = \frac{1}{9} < 1$$

so by Ratio test $\sum_{n=1}^{\infty} (-1)^n \frac{n^2 (n+2)!}{n! \; 3^{2n}}$ conv. absolutely

12 $\displaystyle\sum_{n=1}^{\infty} \left(-\ln\left(e^2 + \frac{1}{n}\right)\right)^{n+1}$

use the Root test to determine if the series converges absolutely or diverges.

$$\lim_{n\to\infty} \sqrt[n]{\left|\left[-\ln\left(e^2+\frac{1}{n}\right)\right]^{n+1}\right|} = \lim_{n\to\infty} \sqrt[n]{\left[\ln\left(e^2+\frac{1}{n}\right)\right]^{n+1}}$$

$$= \lim_{n\to\infty} \left[\ln\left(e^2+\frac{1}{n}\right)\right]^{\frac{n+1}{n}} = \lim_{n\to\infty} \left[\ln\left(e^2+\frac{1}{n}\right)\right]^{1+\frac{1}{n}}$$

Note that as $n\to\infty$, $\frac{1}{n}\to 0$

so $\displaystyle\lim_{n\to\infty} \left[\ln\left(e^2+\frac{1}{n}\right)\right]^{1+\frac{1}{n}} = \ln e^2 = 2 > 1$

so by the Root Test, the series diverges.

16 $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n^{1+n}}$, Root test في السؤال

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n}{n^{1+n}}\right|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^{1+n}}} = \lim_{n\to\infty} \frac{1}{n^{\frac{1+n}{n}}}$$

$$= \lim_{n\to\infty} \frac{1}{n^{1+\frac{1}{n}}} = 0 < 1$$

so by the Root Test, the series converges.

$\boxed{20}$ use any method to determine if the series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{n!}{10^n}$$

use the Ratio Test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{10^{n+1}}}{\frac{n!}{10^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)!}{10^{n+1}} \cdot \frac{10^n}{n!} \right| = \lim_{n\to\infty} \left| \frac{(n+1)\cancel{n!}}{\cancel{10^n}\cdot 10} \cdot \frac{\cancel{10^n}}{\cancel{n!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n+1}{10} \right| = \infty$$

So the series diverges.

$\boxed{28}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-\ln n)^n}{n^n}$ use any ~~test~~ method.

by the Root test, $\displaystyle\lim_{n\to\infty} \sqrt[n]{\left|\frac{(-\ln n)^n}{n^n}\right|} =$

$$\lim_{n\to\infty} \sqrt[n]{\frac{(\ln n)^n}{n^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{(\ln n)^n}}{\sqrt[n]{n^n}} = \lim_{n\to\infty} \frac{\ln n}{n} \overset{\frac{\infty}{\infty}}{=}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0 < 1$$

So the series Converges

$\boxed{38}$ $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(-n)^n}$ use any method.

Ratio test:
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{(-n-1)^{n+1}}}{\frac{n!}{(-n)^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)!}{(-[n+1])^{n+1}} \cdot \frac{(-n)^n}{n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{(n+1)} \cdot \cancel{n!}}{\cancel{(n+1)} \cdot (n+1)^n} \cdot \frac{(n)^n}{\cancel{n!}} \right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^n = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} = \frac{1}{e} < 1$$

So the series converges.

$\boxed{43}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ ~~use~~

Ratio Test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{[(n+1)!]^2}{[2(n+1)]!}}{\frac{(n!)^2}{(2n)!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{[(n+1)!]^2}{[2n+2]!} \cdot \frac{(2n)!}{(n!)^2} \right|$$

$$= \lim_{n\to\infty} \frac{(n+1)\cancel{n!} \cdot (n+1) \cdot \cancel{n!}}{(2n+2)(2n+1)\cancel{(2n)!}} \cdot \frac{\cancel{(2n)!}}{\cancel{n!}\,\cancel{n!}} = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)}$$

$$= \frac{1}{4} < 1$$

So the series converges

$\boxed{46}$ $a_1 = 1, \quad a_{n+1} = \dfrac{1+\tan^{-1} n}{n} a_n$

Ratio test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{1+\tan^{-1} n}{n} \cancel{a_n}}{\cancel{a_n}} \right|$$

$$= \lim_{n\to\infty} \frac{1+\tan^{-1} n}{n} = \frac{1+\frac{\pi}{2}}{\lim_{n\to\infty} n} = 0 < 1$$

So the series converges.

$\boxed{60}$ $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{(2^n)^2}$

The Root Test

$$\lim_{n\to\infty} \sqrt[n]{\left| \frac{n^n}{(2^n)^2} \right|} = \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{(2^2)^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n^n}}{\sqrt[n]{(2^2)^n}}$$

$$= \lim_{n\to\infty} \frac{n}{4} = \infty$$

So the series diverges.