10.6 Alternating Series and Conditional Convergences

The Alternating Series Test :

The series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots$

converges if all three of the following conditions are satisfied:

1- The u_n's are all positive

2- the positive u_n's are (eventually) nonincreasing

$u_n \geqslant u_{n+1}$ for all $n \geqslant N$ for some integer N

3- $\lim_{n \to \infty} u_n = 0$

Def.- A convergent series that is not absolutely convergent is conditionally convergent.

Questions: 8, 13, 20, 25, 30, 39, 42, 50, 54

$\boxed{8}$ Determine if the alternating series converges or diverges.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2+5}{n^2+4}$$

n^{th} term test: $\lim_{n\to\infty} (-1)^{n+1} \cdot \frac{n^2+5}{n^2+4} = DNE$

(Note that $\lim_{n\to\infty} \frac{n^2+5}{n^2+4} = 1$)

So the series diverges.

$\boxed{13}$ Determine if the alternating series converges or diverges.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}+1}{n+1}$$

use the alternating series test:-

$$f(x) = \frac{\sqrt{x}+1}{x+1}$$

$$f'(x) = \frac{(x+1)\cdot \frac{1}{2\sqrt{x}} - (\sqrt{x}+1)\cdot 1}{(x+1)^2} = \frac{x+1-2x-2\sqrt{x}}{2\sqrt{x}\,(x+1)^2}$$

$$= \frac{1-x-2\sqrt{x}}{2\sqrt{x}\,(x+1)^2} < 0 \quad \text{for } x \geq 1$$

$f(x)$ is decreasing

$u_n \geq u_{n+1}$ for $n \geq 1$

$$u_n = \frac{\sqrt{n}+1}{n+1} \geq 0 \quad \text{for } n \geq 1$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{\sqrt{n}+1}{n+1} = \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{1} = \frac{0}{1} = 0$$

So the series ~~diverg~~ converges by A.S.T.

$\boxed{20}$ $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n!}{2^n}$

use the nth term test.

$$\lim_{n\to\infty} \frac{n!}{2^n} = \lim_{n\to\infty} \frac{1.2.3.4\ldots n}{2.2.2.2\ldots 2} \geq \lim_{n\to\infty} \frac{n}{2} = \infty$$

So $\displaystyle\lim_{n\to\infty} \frac{n!}{2^n} = \infty$

So the series diverges.

$\boxed{25}$ Determine if the series converges absolutely, converges or diverges?

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1+n}{n^2}$$

- Converges absolutely?? $\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{1+n}{n^2} \right|$ Converges??

$\sum_{n=1}^{\infty} \frac{1+n}{n^2}$ Converges??

$$\sum_{n=1}^{\infty} \frac{1+n}{n^2} = \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\text{Converges}} + \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\text{diverges}} = \text{diverges.}$$

So the series doesn't converge absolutely.

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1+n}{n^2}$ Converges??

by A.S.T :- ① $u_n = \frac{1+n}{n^2} \geq 0$ for $n \geq 1$

② $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1+n}{n^2} = \lim_{n\to\infty} \frac{1}{2n} = 0$

③ $f(x) = \frac{1+x}{x^2}$, $f'(x) = \frac{x^2 \cdot 1 - (1+x) \cdot 2x}{(x^2)^2} = \frac{x^2 - 2x - 2x^2}{x^4}$

$f'(x) = \frac{-x^2 - 2x}{x^4} < 0$ for $x \geq 1$

So by the A.S.T , the series Converges.

So the series converges conditionaly.

Which of the following converge absolutely, which converge, and which diverge ??

$\boxed{30}$ $\sum_{n=1}^{\infty} (-1)^n \dfrac{\ln n}{n - \ln n}$

- converge absolutely ?? $\sum_{n=1}^{\infty} \dfrac{\ln n}{n - \ln n}$??

Compare with $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges (harmonic series)

$$\lim_{n\to\infty} \frac{\frac{\ln n}{n - \ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n \ln n}{n - \ln n} = \lim_{n\to\infty} \frac{1 + \ln n}{1 - \frac{1}{n}} = \frac{\infty}{1} = \infty$$

So the series $\sum_{n=1}^{\infty} \dfrac{\ln n}{n - \ln n}$ diverges.

$\sum_{n=1}^{\infty} (-1)^n \dfrac{\ln n}{n - \ln n}$??

let ① $u_n = \dfrac{\ln n}{n - \ln n} \geq 0$ for $n \geq 1$ (Note that $n > \ln n$)

② $\lim_{n\to\infty} \dfrac{\ln n}{n - \ln n} = \lim_{n\to\infty} \dfrac{\frac{1}{n}}{1 - \frac{1}{n}} = \dfrac{0}{1-0} = 0$

③ $f(x) = \dfrac{\ln x}{x - \ln x}$, $f'(x) = \dfrac{(x - \ln x) \cdot \frac{1}{x} - \ln x (1 - \frac{1}{x})}{(x - \ln x)^2}$

$$f'(x) = \frac{1 - \cancel{\frac{\ln x}{x}} - \ln x + \cancel{\frac{\ln x}{x}}}{(x - \ln x)^2} = \frac{1 - \ln x}{(x - \ln x)^2}$$

for $x \geq e$; $\ln x \geq 1 \rightarrow 1 - \ln x \leq 0 \rightarrow f'(x) \leq 0$

So the series converges by A.S.T

So the series converges conditionally.

$\boxed{39}$ $\sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{2^n \, n! \, n}$

$\sum_{n=1}^{\infty} \frac{(2n)!}{2^n \, n! \, n}$ converges??

n^{th} term test: $\lim_{n\to\infty} \frac{(2n)!}{2^n \, n! \, n} = \lim_{n\to\infty} \frac{2n(2n-1)\cdot(2n-2)\cdots\cancel{n!}}{2^n \, \cancel{n!} \, n}$

$= \lim_{n\to\infty} \frac{\cancel{2n}(2n-1)(2n-2)\cdots(n+1)}{2^{n-1} \cancel{n}} = \lim_{n\to\infty} \frac{(2n-1)(2n-2)\cdots(n+1)}{2^{n-1}}$

$> \lim_{n\to\infty} \frac{(n+1)^{n-1}}{2^{n-1}} = \lim_{n\to\infty} \left(\frac{n+1}{2}\right)^{n-1} = \infty$

So $\sum_{n=1}^{\infty} \frac{(2n)!}{2^n \, n! \, n}$ diverges

$\sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{2^n \, n! \, n}$

n^{th} term test: $\lim_{n\to\infty} (-1)^n \frac{(2n)!}{2^n \, n! \, n} = DNE$

So $\sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{2^n \, n! \, n}$ diverges

$\boxed{42}$ $\sum_{n=1} (-1)^n \left(\sqrt{n^2+n} - n \right)$

$$\lim_{n\to\infty} \left(\sqrt{n^2+n} - n \right) = \lim_{n\to\infty} \left(\sqrt{n^2+n} - n \right) \cdot \left(\frac{\sqrt{n^2+n} + n}{\sqrt{n^2+n} + n} \right)$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2} + n - \cancel{n^2}}{\sqrt{n^2+n} + n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n} + n}$$

$$= \lim_{n\to\infty} \frac{n}{\sqrt{n^2(1+\frac{1}{n})} + n} = \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}\left(\sqrt{1+\frac{1}{n}} + 1\right)} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}} + 1}$$

$$= \frac{1}{1+1} = \frac{1}{2}$$

So by the nth term test, $\sum_{n=1}^{\infty} \left(\sqrt{n^2+n} - n \right)$ diverges

and by the n^{th} term test, $\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n^2+n} - n \right)$

diverges since $\lim_{n\to\infty} (-1)^n \left(\sqrt{n^2+n} - n \right) = \text{DNE}$

$\boxed{50}$ Estimate the magnitude of the error involved in using the sum of the first four terms to approximate the sum of the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{10^n}$$

$$|E| < |a_5|$$

$$|E| < \left|(-1)^{5+1} \frac{1}{10^5}\right|$$

$$|E| < \left|\frac{1}{10^5}\right| = 0.00001$$

$\boxed{54}$ Determine how many terms should be used to estimate the sum of the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$ with an error of less than 0.001

$$|u_{n+1}| < 0.001 \quad \rightarrow \quad \frac{n+1}{(n+1)^2+1} < \frac{1}{1000}$$

$$1000(n+1) < (n+1)^2 + 1$$

$$1000n + 1000 < n^2 + 2n + 1 + 1$$

$$0 < n^2 - 998n - 998$$

$$n > \frac{998 + \sqrt{(998)^2 - 4(1)(-998)}}{2} \simeq 998.99$$

$$n \geqslant 999$$