$\boxed{37}$ Use the Taylor series generated by e^x at $x = a$ to show that

$$e^x = e^a \left[1 + (x-a) + \frac{(x-a)^2}{2!} + \cdots \right]$$

$$f(a) = e^a$$

$$f'(x) = e^x \rightarrow f'(a) = e^a$$

$$f''(x) = e^x \rightarrow f''(a) = e^a$$

$$f'''(x) = e^x \rightarrow f'''(a) = e^a$$

$$\vdots$$

Taylor series : $f(a) + f'(a)(x - \cancel{f}a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$

$$= e^a + e^a(x-a) + \frac{e^a}{2!}(x-a)^2 + \cdots$$

$$= e^a \left[1 + (x-a) + \frac{(x-a)^2}{2!} + \cdots \right]$$

10.8 Taylor and Maclaurin Series.

- The Taylor series generated by f at $x=a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

- The Maclaurin Series generated by f is:

$$\sum_{k=0}^{\infty} \frac{f^{k}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$$

- Taylor polynomial of order n generated by f at $x=a$ is

$$P_n(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Questions : 3, 14, 20, 22, 27, 32, 37

Pages : 606 - 607

$\boxed{3}$ Find the Taylor polynomials of orders 0, 1, 2 and 3 generated by f at a.

$f(x) = \ln x \quad , a = 1$

$f(1) = \ln(1) = 0$

$f'(x) = \frac{1}{x} \quad \rightarrow \quad f'(1) = 1$

$f''(x) = \frac{-1}{x^2} \quad \rightarrow \quad f''(1) = -1$

$f'''(x) = \frac{2}{x^3} \quad \rightarrow \quad f'''(1) = 2$

$P_0(x) = f(1) = 0$

$P_1(x) = f(1) + f'(1)(x-1) = 0 + 1(x-1) = x - 1$

$P_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2$

$= 0 + 1(x-1) + \frac{-1}{2}(x-1)^2 = x - 1 - \frac{1}{2}(x-1)^2$

$P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$

$= 0 + (x-1) + -\frac{1}{2}(x-1)^2 + \frac{2}{6}(x-1)^3$

$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$

$\boxed{14}$ Find the Maclaurin series for the function

$$f(x) = \frac{2+x}{1-x}$$

Maclaurin series generated by f is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$f'(x) = \frac{3}{(1-x)^2} \rightarrow f'(0) = \frac{3}{(1-0)^2} = 3$$

$$f''(x) = \frac{6}{(1-x)^3} \rightarrow f''(0) = 6$$

$$f'''(x) = \frac{18}{(1-x)^4} \rightarrow f'''(0) = 18$$

$\vdots$

Maclaurin series:

$$= 2 + 3x + \frac{6}{2!}x^2 + \frac{18}{3!}x^3 + \cdots$$

$$= 2 + 3x + 3x^2 + 3x^3 + \cdots$$

$$= 2 + \sum_{n=1}^{\infty} 3x^n$$

$\boxed{20}$ Find the Maclaurin Series for the function $f(x) = \sinh x = \frac{e^{x} - e^{-x}}{2}$

$f(0) = \sinh(0) = \frac{e^{0} - e^{-0}}{2} = 0$

$f'(x) = \cosh(x) = \frac{e^{x} + e^{-x}}{2} \rightarrow f'(0) = \frac{e^{0} + e^{-0}}{2} = 1$

$f''(x) = \sinh(x) \longrightarrow f''(0) = 0$

$f'''(x) = \cosh(x) \longrightarrow f'''(0) = 1$

$\vdots$

Maclaurin series : $f(0) + f'(0)\,x + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}x^3 + \cdots$

$\rightarrow 0 + x + 0 + \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + \cdots$

$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

$\boxed{22}$ Find the Maclaurin series of the function

$$f(x) = \frac{x^2}{x+1}$$

$$f(0) = \frac{0^2}{0+1} = 0$$

$$f'(x) = \frac{x^2+2x}{(x+1)^2} \rightarrow f'(0) = 0$$

$$f''(x) = \frac{2}{(x+1)^3} \rightarrow f''(0) = 2$$

$$f'''(x) = \frac{-6}{(x+1)^4} \rightarrow f'''(0) = -6$$

$\vdots$

Maclaurin Series : $f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$

$$= 0 + 0 + \frac{2}{2!}x^2 + \frac{-6}{3!}x^3 + \cdots$$

$$= x^2 - x^3 + x^4 - x^5 + \cdots$$

$$= \sum_{n=2}^{\infty} (-1)^n x^n$$

$\boxed{27}$ Find the Taylor series generated by f at $x = a$

$$f(x) = \frac{1}{x^2} \quad , \quad a = 1$$

$$f(1) = 1$$

$$f'(x) = \frac{-2}{x^3} \rightarrow f'(1) = -2$$

$$f''(x) = \frac{+6}{x^4} \rightarrow f''(1) = +6$$

$$f'''(x) = -\frac{24}{x^5} \rightarrow f'''(1) = -24$$

Taylor series : $f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!} + \frac{f'''(1)}{3!}(x-1)^3 + \cdots$

$$= 1 + {-2}(x-1) + \frac{6}{2!}(x-1)^2 + \frac{-24}{3!}(x-1)^3 + \cdots$$

$$= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n$$

$\boxed{32}$ Find the Taylor series generated by

$$f(x) = \sqrt{x+1} \quad , \quad a = 0$$

$$f(0) = \sqrt{0+1} = 1$$

$$f'(x) = \frac{1}{2\sqrt{x+1}} \rightarrow f'(0) = \frac{1}{2}$$

$$f''(x) = \frac{-1}{4(x+1)^{3/2}} \rightarrow f''(0) = \frac{-1}{4}$$

$$f'''(x) = \frac{3}{8(x+1)^{5/2}} \rightarrow f'''(0) = \frac{3}{8}$$

Taylor series: $f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \ldots$

$$= 1 + \frac{1}{2}x + \frac{-\frac{1}{4}}{2!}x^2 + \frac{\frac{3}{8}}{3!}x^3 + \ldots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots$$

$\boxed{37}$ Use the Taylor series generated by e^x at $x = a$ to show that

$$e^x = e^a \left[1 + (x-a) + \frac{(x-a)^2}{2!} + \cdots \right]$$

$f(a) = e^a$

$f'(x) = e^x \rightarrow f'(a) = e^a$

$f''(x) = e^x \rightarrow f''(a) = e^a$

$f'''(x) = e^x \rightarrow f'''(a) = e^a$

$\vdots$

Taylor series : $f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$

$$= e^a + e^a(x-a) + \frac{e^a}{2!}(x-a)^2 + \cdots$$

$$= e^a \left[1 + (x-a) + \frac{(x-a)^2}{2!} + \cdots \right]$$

$$= e^x$$