10.9 Convergence of Taylor Series.

* Taylor's Formula:-

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$, for some c between a and x

* If $R_n \to 0$ as $n \to \infty$, then $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$

* The Remainder Estimation Theorem:

If there is a positive constant M such that $|f^{(n+1)}(t)| \le M$ for all t between x and a

then $$|R_n(x)| \le \frac{M\,|x-a|^{n+1}}{(n+1)!}$$

Questions : 10, 12, 18, 28, 35, 37, 41

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$\boxed{10}$ Use substitution to find the taylor series at $x=0$ of the function $\frac{1}{2-x}$

$$\frac{1}{2-x} = \frac{1}{2(1-\frac{x}{2})} = \frac{1}{2}\left(\frac{1}{1-\frac{x}{2}}\right)$$

$$\rightarrow \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad (\text{geometric series } a=1,\ r=x)$$

$$\rightarrow \frac{1}{1-\frac{x}{2}} = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$$

$$\rightarrow \frac{1}{2-x} = \frac{1}{2}\left(\frac{1}{1-\frac{x}{2}}\right) = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}} = \frac{1}{2} + \frac{1}{4}x + \frac{1}{8}x^2 + \cdots$$

$\boxed{12}$ Use power series operations to find the Ta

series at $x=0$ for the function $x^2 \sin x$

Taylor series generated by $\sin x$ at $x=0$ is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

so $$x^2 \sin x = x^2\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+1)!} = x^3 - \frac{x^5}{3!} + \frac{x^7}{5!} - \frac{x^9}{7!} + \cdots$$

$\boxed{18}$ Use power series ~~gen~~ operations to find the Taylor series at $x=0$ ~~to~~ for the function $\sin^2 x$

$$\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\cos(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots$$

$$\sin^2(x) = \frac{1}{2} - \frac{1}{2}\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots\right)$$

$$= \frac{(2x)^2}{2\cdot 2!} - \frac{(2x)^4}{2\cdot 4!} + \frac{(2x)^6}{2\cdot 6!} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2x)^{2n}}{2\cdot(2n)!}$$

$$\sin^2(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, 2^{2n-1}\, x^{2n}}{(2n)!}$$

$\boxed{28}$ use power series operations to find the Taylor series at $x=0$ for the function $\ln(x+1) - \ln(1-x)$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \cdots$$

$$\ln(1+x) - \ln(1-x)$$

$$= \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots\right) - \left(-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \cdots\right)$$

$$= 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{2}{2n+1} x^{2n+1}$$

$\boxed{35}$ Estimate the error if $P_3(x) = x - \frac{x^3}{6}$ is used to estimate the value of $\sin x$ at $x = 0.1$

$$\sin x = x - \frac{x^3}{3!} \Big| + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$|R_n(x)| \le \frac{M\ |x|^{n+1}}{(n+1)!}$$

$$|R_3(x)| \le \frac{M\ |x|^4}{4!}$$

find M, $f^{(4)}(x) = \sin(x) \le 1$ for any $x \in [0, 0.1]$

take $M = 1$

$$|R_3(0.1)| \le \frac{1\ |0.1|^4}{4!} = 4.2 \times 10^{-6}$$

Error $\le 4.2 \times 10^{-6}$

$\boxed{37}$ For approximately what values of x can you replace $\sin x$ by $x - \frac{x^3}{6}$ with an error of magnitute no greater than 5×10^{-4}?

$|E| < 5 \times 10^{-4}$

$\sin x = x - \frac{x^3}{3!} \Big| + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$|E| < \frac{|x|^5}{5!}$ by Alternating Series Estimation

$\frac{|x|^5}{5!} < 5 \times 10^{-4}$

$|x|^5 < 5 \times 10^{-4} \times 5!$

$|x|^5 < 600 \times 10^{-4}$

$|x| < \sqrt[5]{600 \times 10^{-4}} \simeq 0.5697$

$-0.5697 < x < 0.5697$

$\boxed{41}$ The approximation $e^x = 1 + x + \frac{x^2}{2}$ is used when x is small. Use the Remainder Estimation theorem to estimate the error when $|x| < 0.1$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$|R_2(x)| = \left|\frac{f^3(c)\, x^3}{3!}\right| = \left|\frac{e^c\, x^3}{3!}\right| \quad , \; c \text{ between } 0 \text{ and } x$$

$\rightarrow$ $|x| < 0.1 \longrightarrow |x|^3 < (0.1)^3$

$\rightarrow$ if $-0.1 < x < 0$, then $e^c < 1$

$\rightarrow$ $|R_2(x)| < \frac{1\,(0.1)^3}{3!} = 1.67 \times 10^{-4}$

if $0 < x < 0.1 \rightarrow e^c < e^{0.1}$

$$|R_2(0.1)| < \left|\frac{e^{0.1}\,(0.1)^3}{3!}\right| = 1.84 \times 10^{-4}$$

So the error $E < 1.84 \times 10^{-4}$