11.2 Calculus with Parametric Curves.

Parametric formula for $\frac{dy}{dx}$:

if all three derivatives exist and $\frac{dx}{dt} \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

- Parametric formula for $\frac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

- the length of C: $L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$

- Area of surface of Revolution:-

* about x-axis $S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

* about y-axis $S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Questions: 7, 14, 20, 22, 25, 27

pages: 643 - 644

$\boxed{7}$ Find the equation for the tangent to the curve $x = \sec t$, $y = \tan t$ at $t = \frac{\pi}{6}$

Find the value of $\frac{d^2y}{dx^2}$ at this point.

at $t = \frac{\pi}{6} \rightarrow x = \sec\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}}$

$y = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} \rightarrow \left(\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sec^2 t}{\sec t \tan t} = \frac{\sec t}{\tan t}$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{6}} = \frac{\sec \frac{\pi}{6}}{\tan \frac{\pi}{6}} = \frac{\frac{2}{\sqrt{3}}}{\frac{1}{\sqrt{3}}} = 2$$

the tangent: $y - \frac{1}{\sqrt{3}} = 2\left(x - \frac{2}{\sqrt{3}}\right) \rightarrow y = 2x - \sqrt{3}$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\dfrac{\tan t \cdot \sec t \tan t - \sec t \sec^2 t}{\tan^2 t}}{\sec t \tan t}$$

$$= \frac{\cancel{\sec t}\left[\tan^2 t - \sec^2 t\right]}{\cancel{\sec t} \tan^3 t} = \frac{-1}{\tan^3 t}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{6}} = \frac{-1}{\left(\tan\frac{\pi}{6}\right)^3} = \frac{-1}{\left(\frac{1}{\sqrt{3}}\right)^3}$$

$$= \frac{-1}{\frac{1}{3\sqrt{3}}} = -3\sqrt{3}$$

$\boxed{14}$ Find an equation of the line tangent to the curve $x = t + e^t$, ~~at~~ $y = 1 - e^t$, $t = 0$

find $\frac{d^2y}{dx^2}$ at this point ($t=0$).

$\rightarrow$ at $t=0$, $x = 0 + e^0 = 0 + 1 = 1$ $\rightarrow (1, 0)$

$y = 1 - e^0 = 1 - 1 = 0$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-e^t}{1+e^t}$$

$$\left.\frac{dy}{dx}\right|_{t=0} = \frac{-e^0}{1+e^0} = \frac{-1}{1+1} = -\frac{1}{2}$$

the tangent is $y - 0 = -\frac{1}{2}(x-1)$

$$y = \frac{-1}{2}x + \frac{1}{2}$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} =$$

$$\frac{dy'}{dt} = \frac{(1+e^t)\cdot e^t - (-e^t)(e^t)}{(1+e^t)^2} = \frac{-e^t + \cancel{e^{2t}} - \cancel{e^{2t}}}{(1+e^t)^2}$$

$$= \frac{-e^t}{(1+e^t)^2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{-e^t}{(1+e^t)^2}}{1+e^t} = \frac{-e^t}{(1+e^t)^3}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=0} = \frac{-e^0}{(1+e^0)^3} = \frac{-1}{(2)^3} = \frac{-1}{8}$$

$\boxed{20}$ Find the slope of the curve $x = f(t)$, $y = g(t)$ at $t = 0$

* $t = \ln(x-t)$, $y = te^t$

$$1 = \frac{\frac{dx}{dt} - 1}{x - t} \rightarrow x - t = \frac{dx}{dt} - 1$$

$$\frac{dx}{dt} = x - t + 1$$

$$\frac{dy}{dt} = te^{t} + e^{t}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{te^{t} + e^{t}}{x - t + 1}$$

at $t = 0 \rightarrow 0 = \ln(x - 0) \rightarrow 0 = \ln x$

$\rightarrow x = 1$

$$\left.\frac{dy}{dx}\right|_{t=0} = \frac{0(e^{0}) + e^{0}}{1 - 0 + 1} = \frac{0 + 1}{2} = \frac{1}{2}$$

$\boxed{22}$ Find the area enclosed by the y-axis and the curve $x = t - t^2$, $y = 1 + e^{-t}$

$$A = \int_{c}^{d} x \, dy \rightarrow dy = -e^{-t} dt$$

$$A = \int (t - t^2)(-e^{-t}) dt$$

the curve intersect the y-axis when $x = 0$

$t - t^2 = 0 \rightarrow t(1 - t) = 0 \rightarrow t = 0, 1$

$$A = \int_{0}^{1} (t - t^2)(-e^{-t}) dt$$

$$
\begin{array}{ll}
t - t^2 & -e^{-t} \\
1 - 2t & e^{-t} \\
-2 & -e^{-t} \\
0 & e^{-t}
\end{array}
$$

$$A = \left[(t - t^2) e^{-t} - (1 - 2t)(-e^{-t}) + (-2) e^{-t} \right]_0^1$$

$$= \left[\cancel{(1 - 1^2) e^{-1}} + (1 - 2(1))(e^{-1}) - 2e^{-1} \right]$$

$$- \left[\cancel{0\, e^{0}} + (1 - 0) e^{0} - 2 e^{0} \right]$$

$$= -e^{-1} - 2e^{-1} - (1 - 2)$$

$$= -3e^{-1} + 1 = 1 - \frac{3}{e}$$

$\boxed{25}$ Find the length of the curve

$x = \cos t \,,\; y = t + \sin t \,, \quad 0 \le t \le \pi$

$$L = \int_0^\pi \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \; dt$$

$$\frac{dx}{dt} = -\sin t$$

$$\frac{dy}{dt} = 1 + \cos t$$

$$L = \int_0^\pi \sqrt{(-\sin t)^2 + (1 + \cos t)^2} \; dt$$

$$= \int_0^\pi \sqrt{\sin^2 t + \cos^2 t + 2\cos t + 1} \; dt$$

$$= \int_0^\pi \sqrt{1 + 2\cos t + 1} \; dt = \int_0^\pi \sqrt{2 + 2\cos t} \; dt$$

$$= \sqrt{2} \int_0^\pi \sqrt{(1 + \cos t) \cdot \frac{(1 - \cos t)}{(1 - \cos t)}} \; dt$$

$$= \sqrt{2} \int_0^\pi \sqrt{\frac{1 - \cos^2 t}{1 - \cos t}} \; dt = \sqrt{2} \int_0^\pi \sqrt{\frac{\sin^2 t}{1 - \cos t}} \; dt$$

$$L = \sqrt{2} \int_0^{\pi} \frac{\sin t}{\sqrt{1-\cos t}} \, dt$$

let $u = 1-\cos t$

$du = \sin t \, dt$

$$= \sqrt{2} \int \frac{du}{\sqrt{u}} = \sqrt{2} \frac{u^{\frac{1}{2}}}{\frac{1}{2}}$$

$$= 2\sqrt{2} \sqrt{u} = 2\sqrt{2} \sqrt{1-\cos t} \Big|_0^{\pi}$$

$$= 2\sqrt{2} \left[\sqrt{1-\cos \pi} - \sqrt{1-\cos 0} \right]$$

$$= 2\sqrt{2} \left[\sqrt{2} - 0 \right] = 2\sqrt{2}(\sqrt{2}) = 4$$

$\boxed{27}$ Find the length of the curve

$$x = \frac{t^2}{2}, \quad y = \frac{(2t+1)^{3/2}}{3}, \quad 0 \leq t \leq 4$$

$$\frac{dx}{dt} = t, \quad \frac{dy}{dt} = \frac{1}{\cancel{3}} \left(\frac{\cancel{3}}{\cancel{2}}\right) (2t+1)^{\frac{1}{2}} (\cancel{2}) = \sqrt{2t+1}$$

$$L = \int_0^4 \sqrt{(t)^2 + (\sqrt{2t+1})^2} \, dt = \int_0^4 \sqrt{t^2+2t+1} \, dt$$

$$= \int_0^4 \sqrt{(t+1)^2} \, dt = \int_0^4 |t+1| \, dt$$

$$L = \int_0^4 t + 1 \, dt$$

$$= \left[\frac{t^2}{2} + \cancel{8}t\right]\Big|_0^4$$

$$= \frac{16}{2} + 4 - (0 + 0) = 8 + 4 = 12$$