11.3 Polar Coordinates

* $P(r, \theta)$

  r: Directed distance from O origin (pole) to P

  θ: Directed angle from initial ray to OP

Equation	Graph
$r = a$	circle of radius $\lvert a \rvert$ centered at O
$\theta = \theta_0$	line through O making an angle θ_0 with the initial ray.

* Equations Relating Polar and Cartesian Coordinates.

  $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$

  $\tan\theta = \frac{y}{x}$

[1] Which polar coordinate pairs label the same point?

a) $(3,0)$ b) $(-3,0)$ c) $(2,\frac{2\pi}{3})$

d) $(2,\frac{7\pi}{3})$ e) $(-3,\pi)$ f) $(2,\frac{\pi}{3})$

g) $(-3,2\pi)$ h) $(-2,-\frac{\pi}{3})$

$(3,0) \longleftrightarrow (-3,\pi)$

$(-3,0) \longleftrightarrow (-3,2\pi)$

$(2,\frac{2\pi}{3}) \longleftrightarrow (-2,-\frac{\pi}{3})$

$(2,\frac{7\pi}{3}) \longleftrightarrow (2,\frac{\pi}{3})$

[6] Find the Cartesian coordinates of the following points

a) $(\sqrt{2},\frac{\pi}{4})$

$$\left.\begin{aligned} x &= \sqrt{2}\cos\tfrac{\pi}{4} = \sqrt{2}\cdot\tfrac{1}{\sqrt{2}} = 1 \\ y &= \sqrt{2}\sin\tfrac{\pi}{4} = \sqrt{2}\cdot\tfrac{1}{\sqrt{2}} = 1 \end{aligned}\right\} \rightarrow (1,1)$$

b) $(1, 0)$

$$\left.\begin{aligned} x &= 1 \cos 0 = 1 \\ y &= 1 \sin 0 = 0 \end{aligned}\right\} \rightarrow (1, 0)$$

c) $(0, \frac{\pi}{2})$

$$\left.\begin{aligned} x &= 0 \cos(\tfrac{\pi}{2}) = 0 \\ y &= 0 \sin(\tfrac{\pi}{2}) = 0 \end{aligned}\right\} \rightarrow (0, 0)$$

d) $(-\sqrt{2}, \frac{\pi}{4})$

$$\left.\begin{aligned} x &= -\sqrt{2} \cos\tfrac{\pi}{4} = -1 \\ y &= -\sqrt{2} \sin\tfrac{\pi}{4} = -1 \end{aligned}\right\} \rightarrow (-1, -1)$$

e) $(-3, \frac{5\pi}{6})$

$$\left.\begin{aligned} x &= -3 \cos\left(\tfrac{5\pi}{6}\right) = -3 \cdot -\tfrac{\sqrt{3}}{2} = \tfrac{3\sqrt{3}}{2} \\ y &= -3 \sin\left(\tfrac{5\pi}{6}\right) = -3 \cdot +\tfrac{1}{2} = -\tfrac{3}{2} \end{aligned}\right\} \rightarrow \left(\tfrac{3\sqrt{3}}{2}, -\tfrac{3}{2}\right)$$

f) $(5, \tan^{-1}(\frac{4}{3}))$ بالآلة الحاسبة

$$\left.\begin{aligned} x &= 5 \cos\left(\tan^{-1}\tfrac{4}{3}\right) = 5\,(0.6) = 3 \\ y &= 5 \sin\left(\tan^{-1}\tfrac{4}{3}\right) = 5\,(0.79968) \cong 4 \end{aligned}\right\} \quad (3, 4)$$

g) $(-1, 7\pi)$

$x = -\cos(7\pi) = -(-1) = 1$

$y = -\sin(7\pi) = 0$

$\} \rightarrow (1, 0)$

b) $(2\sqrt{3}, \frac{2\pi}{3})$

$x = 2\sqrt{3}\cos(\frac{2\pi}{3}) = 2\sqrt{3}(-\frac{1}{2}) = -\sqrt{3}$

$y = 2\sqrt{3}\sin(\frac{2\pi}{3}) = 2\sqrt{3}(\frac{\sqrt{3}}{2}) = 3$

$\} \rightarrow (-\sqrt{3}, 3)$

7 Find the polar coordinates, $0 \leq \theta < 2\pi$, $r \geq 0$ of the following points

a) $(1, 1)$

$r = \sqrt{1^2 + 1^2} = \sqrt{2}$

~~$\tan\theta = \frac{1}{1} \rightarrow$~~

$\sin\theta = \frac{1}{\sqrt{2}}$, $\cos\theta = \frac{1}{\sqrt{2}} \rightarrow \theta = \frac{\pi}{4}$

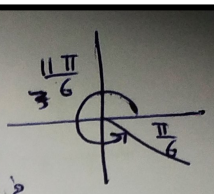
$\rightarrow (\sqrt{2}, \frac{\pi}{4})$

b) $(-3, 0)$, $r = \sqrt{(-3)^2 + 0^2} = 3$

$\sin\theta = 0$, $\cos\theta = -1 \rightarrow \theta = \pi$

$\rightarrow (3, \pi)$

c) $(\sqrt{3}, -1)$



$r = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$

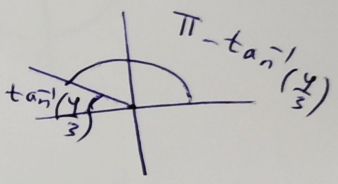
$\sin\theta = -\frac{1}{2}$, $\cos\theta = \frac{\sqrt{3}}{2}$ → في الربع الرابع $\theta = \frac{11\pi}{6}$

$\rightarrow \left(2, \frac{11\pi}{6}\right)$

d) $(-3, 4)$

$r = \sqrt{(-3)^2 + (4)^2} = 5$

$\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{-3}{5}$ الربع الثاني



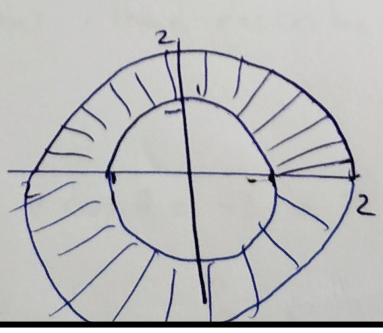
$\tan\theta^* = \frac{4}{3} \rightarrow \theta^* = \tan^{-1}\left(\frac{4}{3}\right)$

$\rightarrow \left(5, \pi - \tan^{-1}\left(\frac{4}{3}\right)\right)$

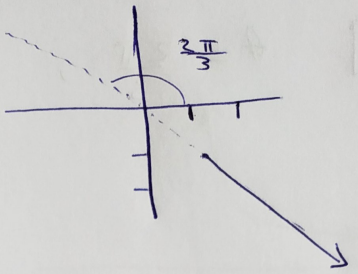
note that $\tan^{-1}\left(\frac{4}{3}\right) \simeq 59$

Graph the sets of points whose polar coordinates satisfy the equations and inequalities:-

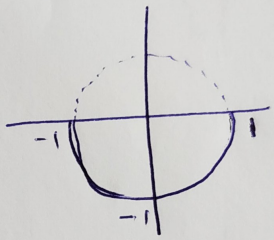
14 $1 \leq r \leq 2$



16 $\theta = \frac{2\pi}{3}$, $r \leq -2$

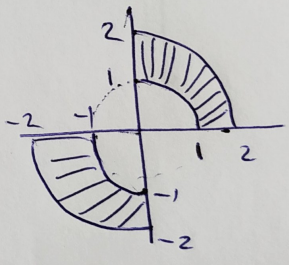


22 $0 \leq \theta \leq \pi$, $r = -1$



26 $0 \leq \theta \leq \frac{\pi}{2}$, $1 \leq |r| \leq 2$

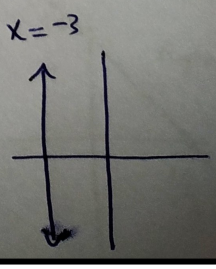
$1 \leq r \leq 2$ or $-2 \leq r \leq -1$



32 Replace the polar equations with equivalent Cartesian equations. Then describe the graph.

$r = -3 \sec\theta$

$r = \frac{-3}{\cos\theta} \rightarrow r\cos\theta = -3 = x$



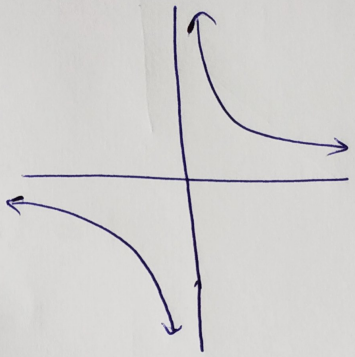
vertical line through $(-3, 0)$

38 $r^2 \sin 2\theta = 2$

$2r^2 \sin\theta \cos\theta = 2$

$(r\sin\theta)(r\cos\theta) = 1$

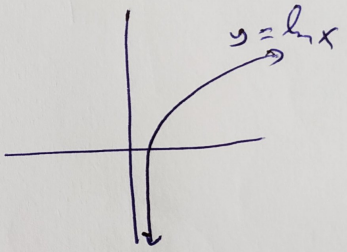
$xy = 1 \rightarrow y = \frac{1}{x}$



42 $r\sin\theta = \ln r + \ln\cos\theta$

$r\sin\theta = \ln r\cos\theta$

$y = \ln x$



52 $r\sin\left(\frac{2\pi}{3} - \theta\right) = 5$

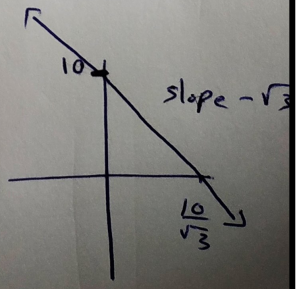
$r\left[\sin\left(\frac{2\pi}{3}\right)\cos\theta - \cos\left(\frac{2\pi}{3}\right)\sin\theta\right] = 5$

$r\left[\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta\right] = 5$

$\frac{\sqrt{3}}{2} r\cos\theta + \frac{1}{2} r\sin\theta = 5$

$\frac{\sqrt{3}}{2} x + \frac{1}{2} y = 5$

$\sqrt{3}x + y = 10$



62 $x^2 + xy + y^2 = 1$

$x^2 + y^2 + xy = 1$

$r^2 + (r\cos\theta)(r\sin\theta) = 1$

$r^2 + r^2 \cos\theta \sin\theta = 1$

$r^2(1 + \cos\theta \sin\theta) = 1$