10.2 : series

- **Def**: A serie is an infinit sum of the form

$$a_1 + a_2 + \cdots + a_n = \sum_{n=0}^{\infty} a_n = S_n$$

- **Partial sums of a series**

$S_1 = a_1$
$S_2 = a_1 + a_2$
$S_3 = a_1 + a_2 + a_3$
$S_n = a_1 + a_2 \cdots + a_n$

$$\{S_n\} = S_1, S_2, S_3 \dots S_n$$

Note: $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} n = \frac{n(n+1)}{2}$

- **Convergane and divergance of series**

- The series $\sum a_n$ Converges to L if the sequence S_n Converges to L [L is the sum]

- The series $\sum a_n$ diverges if the sequence S_n diverges

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- **Geometric series**

Def $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$

$r = \text{ratio} = \frac{ar^n}{ar^{n-1}}$, a = first term

$\sum_{n=1}^{\infty} ar^{n-1} \ (a \neq 0)$ → **Converges** to $\frac{a}{1-r}$ **if** $|r| < 1$
→ **diverges if** $|r| \geq 1$

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• Telescoping series

Def:- A series whose partial sums eventually only have a fixed number of terms after cancellation

Example:- $\sum_{n=0}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$

$$= \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) \cdots \left(\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$$S_n = 1 - \frac{1}{n+1}$$

• To know the sum, we take $\lim_{n\to\infty}$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$$

→ And The series converges

Note:- * If $\sum_{n=10}^{\infty} a_n$ converges and we add (or deleting) terms to it (infinte terms) It remains converge

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_3 \cdots a_9 + \sum_{n=10} a_n$$

Theories of series

if $\sum a_n = A$, $\sum b_n = B$ } Converge

- Then $\sum (a_n \pm b_n)$ Converges to $A \pm B$
- $\sum k a_n$ Converges to kA

If $\sum a_n$ Converges and $\sum b_n$ diverges

- $\sum a_n \pm b_n$ diverges
- $\sum k b_n$ diverges if $k \neq 0$

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- Tests to Know whither the series is Converge or diverge

1) nth test terms

- $\sum a_n$ diverges
If $\lim_{n\to\infty} a_n$ DNE or $\neq 0$ or ∞ or $-\infty$

- test fails :-
If $\lim a_n = 0$
$\Rightarrow$ Then you try other tests

10.3 :-

2) Integral test

If $\sum a_n$ is a series with $a_n \geq 0$ $\forall n \geq N$ and If $f(x)$ is a Continuous, Positive and decreasing function with $f(n) = a_n$ Then

$$\int_N^\infty f(x)\,dx \text{ and } \sum_{n=N}^\infty a_n \text{ behave the same}$$

(both Converges or both diverges)

Note :- $\sum \frac{1}{n}$ harmonic series

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10.4

3) Comparison test

- If $\sum a_n$, $\sum b_n$, $\sum c_n$

$a_n , b_n , c_n \geq 0 \quad \forall n \geq N$

and $a_n \leq b_n \leq c_n$

- If $\sum c_n$ Converges Then $\sum b_n$ Conv
- If $\sum a_n$ diverges Then $\sum b_n$ diverges

4) limit Comparison test

If $a_n , b_n \geq 0 \quad \forall n \geq N$:-

→ If $\lim\limits_{n \to \infty} \frac{a_n}{b_n} = C$, $C > 0$

Then $\sum a_n$, $\sum b_n$ both Converge or both diverge

→ If $\lim\limits_{n \to \infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ Converge Then $\sum a_n$ Converge

→ If $\lim\limits_{n \to \infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ div Then $\sum a_n$ div

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10.5:-

5) Ratio test

- Given $\sum a_n$, $a_n \geq 0$

let $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$

$\rho < 1 \rightarrow$ series converges

$\rho > 1 \rightarrow$ series diverges

$\rho = 1$ Test fails

6) nth Root test

let $a_n \geq 0$

$\sum a_n$, let $\rho = \lim_{n\to\infty} \sqrt[n]{a_n}$

$\rho < 1$ converges

$\rho > 1$ diverges

$\rho = 1$ test fails

* If $\rho = 1$ and if:-

$\frac{a_{n+1}}{a_n} > 1$ and $a_1 > 0$

- nth term test tells us that it diverges

but if:

$\frac{a_{n+1}}{a_n} < 1$ we have no info

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