

10.2 : series

- **Def:** A serie is an infinit sum of the form

$$a_1 + a_2 + \dots + a_n = \sum_{n=0}^{\infty} a_n = S_n$$

- **Partial sums of a series**

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$S_n = a_1 + a_2 + \dots + a_n$$

$$\{S_n\} = S_1, S_2, S_3, \dots, S_n$$

Note: $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} n = \frac{n(n+1)}{2}$

- **Convergence and divergence of series**

• The series $\sum a_n$ Converges to L if the sequence S_n Converges to L [L is the sum]

• The series $\sum a_n$ diverges if the sequence S_n diverges

- **Geometric series**

Def $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots + ar^{n-1} + \dots$

$r = \text{ratio} = \frac{ar^n}{ar^{n-1}}$, $a = \text{first term}$

$\sum_{n=1}^{\infty} ar^{n-1}$ ($a \neq 0$) $\begin{cases} \text{Converges to } \frac{a}{1-r} \text{ if } |r| < 1 \\ \text{diverges if } |r| \geq 1 \end{cases}$

Alaa Ekaiwi

Telescoping series

Def: A series whose partial sums eventually only have a fixed number of terms after cancellation

Example:
$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$= \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$S_n = 1 - \frac{1}{n+1}$$

To know the sum, we take $\lim_{n \rightarrow \infty}$

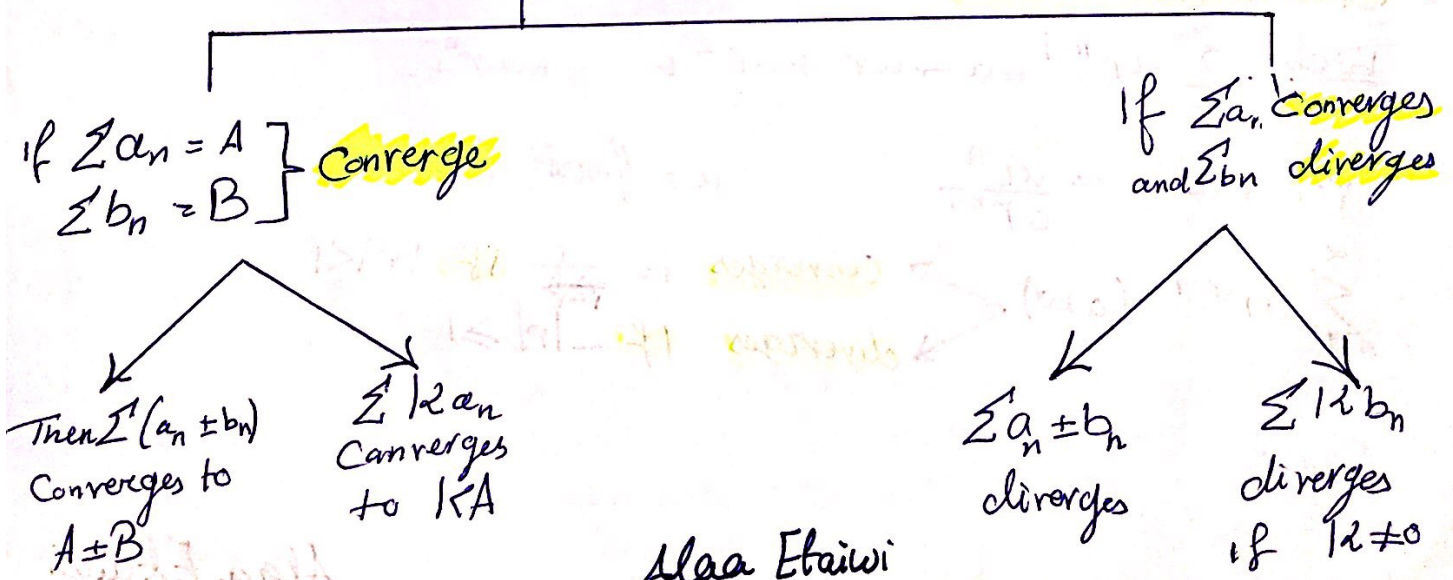
$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1} \right) = 1$$

→ And the series converges

Note: * If $\sum_{n=10}^{\infty} a_n$ converges and we **add** terms to it (infinite terms) it **remains convergent** or **deleting**

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_q + \sum_{n=10}^{\infty} a_n$$

Theories of series



Alaa Etaiwi

- Tests to know whether the series is converge or diverge

1) nth term test

• $\sum a_n$ diverges
if $\lim_{n \rightarrow \infty} a_n$ DNE
or $\neq 0$
or ∞ or $-\infty$

• test fails :-
if $\lim_{n \rightarrow \infty} a_n = 0$

\Rightarrow Then you try other tests

10.3 :-

2) Integral test

If $\sum a_n$ is a series with $a_n \geq 0 \quad \forall n \geq N$ and
if $f(x)$ is a continuous, positive and decreasing
function with $f(n) = a_n$ Then

$\int_N^{\infty} f(x) dx$ and $\sum_{n=N}^{\infty} a_n$ behave the same

(both converges or both diverges)

Note :- $\sum \frac{1}{n}$ harmonic series

Alaa Etaiwi

10.4

3) Comparison test

• If $\sum a_n, \sum b_n, \sum c_n$

$$a_n, b_n, c_n \geq 0 \quad \forall n \geq N$$

$$\text{and } a_n \leq b_n \leq c_n$$

• If $\sum c_n$ Converges
Then $\sum b_n$ Conv

• If $\sum a_n$ diverges
Then $\sum b_n$ diverges

4) Limit Comparison test

If $a_n, b_n \geq 0 \quad \forall n \geq N$:-

→ If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c, c > 0$

Then $\sum a_n, \sum b_n$ both **Converge** or both **diverge**

→ If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$

and $\sum b_n$ Converge Then

$\sum a_n$ Converge

→ If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ div Then $\sum a_n$ div

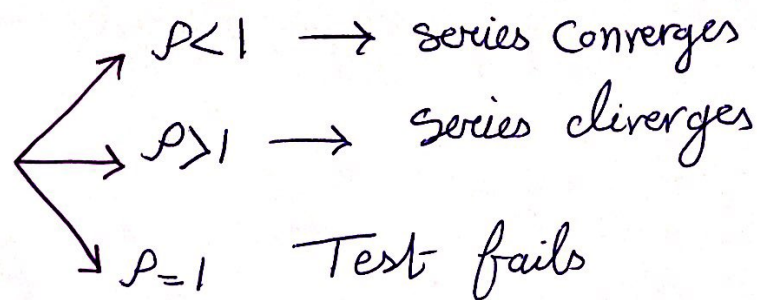
Alaa EbaWi

10.5:-

5) Ratio test

- Given $\sum a_n$, $a_n \geq 0$

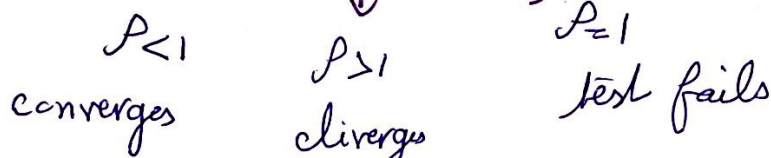
$$\text{let } \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \rho$$



6) nth Root test

let $a_n \geq 0$

$$\sum a_n, \text{ let } \rho = \lim_{n \rightarrow \infty} \sqrt[n]{a_n}$$



* If $\rho = 1$ and if

$$\frac{a_{n+1}}{a_n} > 1 \text{ and } a_n > 0$$

• nth term test tells us that it diverges

but if:

$$\frac{a_{n+1}}{a_n} < 1 \text{ we have no info}$$

Maa Etaiwi