Taylor series & Maclaurin series

- $f(x)$ is a differentiable function for all orders around a center $x=a$

The taylor series is :-

This is the derivative order

$$T.S = f(a) + f'(a)\frac{(x-a)^1}{1!} + f^{(2)}(a)\frac{(x-a)^2}{2!} + \dots + f^{(n)}(a)\frac{(x-a)^n}{n!} + \dots$$

nth derivative at $x=a$

The Maclaurin series occurs when $a=0$:-

$$M.S = f(0) + f'(0)\frac{(x)}{1!} + f^2(0)\frac{(x)^2}{2!} + \dots + f^n(0)\frac{(x)^n}{n!} + \dots$$

- To know wheather a Taylor series conv or div ? You use the technique in 10.7 (Power series) or if it was a Geometric series you use r

- **Note :-**

The function $f(x)$ equals the Taylor series only in the interval of convergance

Taylor Polynomial :- $P_n(x) = f(a) + f'(a)\frac{(x-a)}{1!} + \dots f^n(a)\frac{(x-a)^n}{n!} + \dots$

Alaa Etaiwi

Things that you have to know in this section

• You should know these T.series by heart :-

$\rightarrow e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \sum_{n=0}^{\infty} \frac{x^n}{n!} \cdots$ around $a=0$, $-\infty < x < \infty$

$\rightarrow \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$ around $a=0$, $-\infty < x < \infty$

$\rightarrow \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$ around $a=0$, $-\infty < x < \infty$

$\rightarrow \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} + \cdots$ where $|x|<1$ around $a=0$

$\rightarrow \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \sum_{n=0}^{\infty} (-1)^n x^n + \cdots$ $-1 < x < 1$

$\rightarrow \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \sum_{n=0}^{\infty} x^n + \cdots$ $-1 < x < 1$

$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$ $-1 \leq x \leq 1$

• Steps of finding a Taylor series :-

• Exp : find the Taylor series generated by $f(x) = \frac{1}{x}$ at $a=2$

Step 1 1- find the first 4 or 5 derivatives

$f(x) = \frac{1}{x}$ $\quad f(2) = \frac{1}{2}$

negative when $n=1$ $\rightarrow$ $f'(x) = \frac{-1}{x^2}$ $\quad f'(2) = \frac{-1}{4}$

positive when $n=2$ $\rightarrow$ $f^{(2)}(x) = \frac{2}{x^3}$ $\quad f^{2}(2) = \frac{2}{8} = \frac{1}{4}$

so $(-1)^n$

$f^{(3)}(x) = \frac{-2.3}{x^4}$ $\quad f^{3}(2) = \frac{-2.3}{2^4}$

$f^{4}(x) = \frac{2.3.4}{x^5}$ $\quad f^{4}(2) = \frac{2.3.4}{2^5}$

Step 2 $f^{n}(x) = \frac{(-1)^n 2.3.4 \cdots n}{x^{n+1}}$ $\quad f^{n}(2) = \frac{(-1)^n \overbrace{2.3.4 \cdots n}^{n!}}{2^{n+1}}$

Alaa Etaiwi

Step 3:- set up ...

Now : $T.S = f(2) + f'(2)\frac{(x-2)}{1} + f^{2}(2)\frac{(x-2)^2}{2!} + \cdots f^{n}(2)\frac{(x-2)^n}{n!} + \cdots$

$= \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{\cancel{2}(x-2)^2}{(2^3)(\cancel{2})} + \cdots \frac{(-1)^n \cancel{n!}}{2^{n+1}}\frac{(x-2)^n}{\cancel{n!}} \cdots$

$$\boxed{T.S = \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} + \cdots \frac{(-1)^n (x-2)^n}{2^{n+1}} \cdots}$$

It's a G.S

To know if conv or div:-

$r = -\frac{(x-2)}{2}$

$|r| < 1 \rightarrow$ Conv

so $\boxed{0 < x < 4}$

Result:- $\frac{1}{x} = \frac{1}{2} - \frac{(x-2)}{2^2} + \cdots \cdots$ when $0 < x < 4$

- If he asked for the Mac. series of the function $\frac{1}{1+x}$

$M.S = f(0) + f'(0)\frac{x}{1} + f^{2}(0)\frac{x^2}{2!} - \cdots f^{n}(0)\frac{x^n}{n!} + \cdots$

Step 1:- $f(x) = \frac{1}{1+x}$ $\qquad f(0) = 1$

$f'(x) = \frac{-1}{(1+x)^2}$ $\qquad f'(0) = -1$

$f^{2}(x) = \frac{2}{(1+x)^3}$ $\qquad f^{2}(0) = 2$

$f^{3}(x) = \frac{-2.3}{(1+x)^4}$ $\qquad f^{3}(0) = -2.3$

Step 2:- $M.S = 1 - x + \frac{\cancel{2}x^2}{\cancel{2!}} - \frac{\cancel{2.3}}{\cancel{3!}}x^3 - \cdots (-1)^n \frac{\cancel{n!}\, x^n}{\cancel{n!}} + \cdots$

$$\boxed{M.S = 1 - x + x^2 - x^3 \cdots (-1)^n x^n + \cdots}$$

Alaa Etaiwi