Convergance of Taylor series

Theory :- Taylor formula : There exist $c \in$ Interval of differen where c is between a and x such that :-

$$f(x) = P_n(x) + R_n(x) \quad \text{Remainder (error)} = \frac{f^{(n+1)}(c)(x-a)^n}{(n+1)!}$$

$$f(a) + f'(a)(x-a) ------ + \frac{f^n(a)(x-a)^n}{n!}$$

And if $\lim R_n(x) = 0$ Then T.S Converges to $f(x)$

$\rightsquigarrow$ usually you prove that using sandwich Theorrem

Theory :- Constant M And Remainder Estimation Th

if there exists a Constant M such that

$$|f^n(x)| \leq M \quad \text{for} \quad x \in I$$

then $$|R_n(x)| \leq \frac{M(x-a)^{n+1}}{(n+1)!}$$

In this case :- $\lim_{n \to \infty} R_n(x) = 0$ and T.S converg to $f(x)$

Estimating Errors

- You can estimate errors using :-

→ Alternating.S Estimation Theorem . A.S.E.T

or

→ Remainder Estimation Theorem R.E.T

Alaa Etaiwi

How to solve Problems Concerning T.S , M.S , convergance and Errors

A: T.S / M.S / Conv

- for example: if you want to find Maclaurin series of e^x :-

↳ **step 1:- find derivatives of e^x**

$f(x) = e^x$
$f'(x) = e^x$
⋮
$f^n(x) = e^x$

Note:-
you can use $R_n(x)$ Theory to prove that T.S converges to $f(x)$ knowing that $\lim_{n\to\infty} R_n(x) = 0$

↳ **step 2:- plug $x = \underset{\text{center}}{a}$ in $f^n(x)$**

$f^n(c) = f^n(0) = e^0 = 1$

↳ **step 3:- find $\sum$:-**

$$\sum_0^\infty \frac{f^n(0)\, x^n}{n!} = \sum_0^\infty \frac{x^n}{n!}$$

↳ **step 4:- find interval of Convergance**

We can use Ratio test :-

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{n+1} |x| = 0 < 1$$

∴ So it converges abs for all x ($x \in (-\infty, \infty)$)

↳ **step 5 : represent your series**

$$f(x) = e^x = \sum_0^\infty \frac{x^n}{n!} \Rightarrow e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \text{------}$$

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M. series of Cos X & Sin X :-

Cos X :-

1- $f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$f''''(x) = \cos x$

2) • f^n here has two cases :- order → even / → odd

→ If order is even $(\Rightarrow 2n)$ Then $f^{2n}(x) = (-1)^n \cos x$ ← $x=0$ so it's always 1

→ If order is odd $(\Rightarrow 2n+1)$ Then $f^{2n+1}(x) = (-1)^{n+1} \sin x$ ← since $x=0$ it's always zero

so :-

3) $$\Longrightarrow \sum_{n=0}^{\infty} \frac{(-1)^n \overset{=1}{\cos x}\, X^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n X^{2n}}{(2n)!}$$

4) to find interval of convergance :- let's use Ratio test

$$\lim_{n\to\infty} \left| \frac{X^{2n+2}}{(2n+2)!} \cdot \frac{2n!}{X^{2n}} \right| = \lim_{n\to\infty} \frac{|X|^2}{(2n+2)(2n+1)} = 0 < 1 \quad \text{so conv for all } x$$

5) our series :-

$$f(x) = \cos X = 1 - \frac{X^2}{2} + \frac{X^4}{4!} - \frac{X^6}{6!} + \frac{X^8}{8!} + \cdots + \frac{(-1)^n X^{2n}}{(2n)!} + \cdots$$

Note → Know that the 0's of the odd order does exist :-

$$\cos X = 1 + 0 - \frac{X^2}{2} + 0 + \frac{X^4}{4!} - \cdots$$

so here $P_4(x) = 1 - \frac{X^2}{2} + \frac{X^4}{4!}$

• for **Sin X** it's the same Way

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Estimating Errors :-

*- If the Question Gives you an order $P_n(x)$ / the function / and value at x :- & wants Error

Ex : Exer 35/36/ in the book page 595

let's solve 36 :-

$P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$

at $x = \frac{1}{2}$

1- first c is between x and $a = 0$ (constant; center)

$0 < c < \frac{1}{2}$

Take e for all :-

$e^0 < e^c < e^{\frac{1}{2}}$

$1 < \underline{e^c < e^{\frac{1}{2}}}$

$R_4(x) = \frac{f^5(c)\, x^5}{5!}$

$R_4(x) = \frac{e^c\, x^5}{5!}$

$\frac{x^5}{5!} * \left(e^c < e^{\frac{1}{2}}\right)$

$\left|\frac{e^c\, x^5}{5!}\right| < \frac{e^{\frac{1}{2}}\, x^5}{5!}$ $\quad x = \frac{1}{2}$ (Given)

$R_4(x)$

$$\boxed{R_4(x) < \frac{e^{\frac{1}{2}}\left(\frac{1}{2}\right)^5}{5!}}$$

error $< \frac{e^{\frac{1}{2}}\left(\frac{1}{2}\right)^5}{5!}$

• Note :
if the T.S is Alternating use A.S.E.T

* If the Question gives you the maximum value of error / The function and it's replacement and wants x (You use A.SET) ∴ the series is Altern.

Ex:- Exer 37

$\sin x = x - \frac{x^3}{6} \Big| + \frac{x^5}{5!}$

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$|\text{Error}| < \underbrace{5 \times 10^{-4}}_{\text{Given}}$

$\left|\frac{x^5}{5!}\right| < 5 \times 10^{-4} \Rightarrow \sqrt[5]{-6 \times 10^{-3}} < x < \sqrt[5]{6 \times 10^{-3}}$

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* If the Question gives you the function and it's replacement and X as a range :- and wants Error.

Ex: Exer 38/39/40,

let's solve Exer 38:-

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!}$$

A.S.E.T :-

$$|Error| < \frac{x^4}{4!}$$

$$-0.5 < x < 0.5$$

$$0 < x^4 < (0.5)^4$$

$$\frac{x^4}{4!} < \frac{(0.5)^4}{4!}$$

$$|E| < \frac{(0.5)^4}{4!}$$

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