Calculus with Parametric Curves

If you have $x = f(t)$, $y = g(t)$ $\quad t \in I$

Then the derivative will be :-

نشتقها

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

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The Second derivative is :-

$$y'' = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt}$$

length of Curve :- (Parametric)

- Reminder $\quad L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$x = f(t) \qquad y = g(t) \qquad a \leq t \leq b$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

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How to Solve Problems
(Examples)

- **finding first & second derivatives**

If :- $x = t - t^2$ and $y = t - t^3$:-

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{1-2t}$$

$$y'' = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt} = \frac{\frac{(1-2t)(-6t) - (1-3t^2)(-2)}{(1-2t)^2}}{(1-2t)}$$

$$= \frac{6t^2 - 6t + 2}{(1-2t)^3}$$

- **finding Tangent** (optaining $\frac{dy}{dx}$)

Ex : $x^3 + 2t^2 = 9$ $\qquad 2y^3 - 3t^2 = 4$ $\qquad$ at $t = 2$

differentiate x and y to t :-

$$3x^2 \frac{dx}{dt} + 4t = 0 \qquad 6y^2 \frac{dy}{dt} - 6t = 0$$

$$\left.\frac{dx}{dt}\right|_{t=2} = \frac{-4t}{3x^2} \qquad \left.\frac{dy}{dt}\right|_{t=2} = \frac{6t}{6y^2} = \frac{t}{y^2}$$

at $t = 2$:- $\quad x^3 + 2t^2 = 9$ $\quad$ and $\quad 2y^3 - 3t^2 = 4$

$x^3 + 8 = 9 \rightarrow \boxed{x = 1}$ $\qquad 2y^3 = 16 \Rightarrow \boxed{y = 2}$

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Now :-

$$\left.\frac{dx}{dt}\right|_{\substack{t=2\\x=1}} = -\frac{4x^2}{3} = \frac{-8}{3} \quad \text{and} \quad \left.\frac{dy}{dt}\right|_{\substack{t=2\\y=2}} = \frac{2}{4} = \frac{1}{2}$$

→ Knowing that :- $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$$\left.\frac{dy}{dx}\right|_{t=2} = \frac{\frac{1}{2}}{\frac{-8}{3}} = \frac{-3}{16}$$

• **Area** → Find Area under $y = x^3$ over $[0, 1]$

By substituting

Ex :- $x = t^2 \quad y = t^6 \quad 0 \le t \le 1$

$$A = \int_0^1 y\,dx = \int_0^1 t^6 (2t\,dt)$$

$$= \int_0^1 2t^7\,dt = \left.\frac{2t^8}{8}\right]_0^1$$

$$= \frac{2}{8} = \frac{1}{4}$$

• **length of Curves**

$x = \cos t$ and $y = \sin t + t \quad 0 \le t \le \pi$

find L :-

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^\pi \sqrt{(-\sin t)^2 + (\cos t + 1)^2}\,dt$$

$$= \int_0^\pi \sqrt{\sin^2 t + \cos^2 t + 1 + 2\cos t}\,d$$

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$$= \sqrt{2} \int_0^{\pi} \sqrt{1 + \cos\theta} \; dt \qquad \cos\theta = 2\cos^2\frac{\theta}{2} - 1$$

$$= \sqrt{2} \int_0^{\pi} \sqrt{2\cos^2\frac{\theta}{2}} \; d\theta$$

$$= 2 \int_0^{\pi} \left|\cos\frac{\theta}{2}\right| dt \qquad 0 \leq \cos\frac{\theta}{2} \quad \text{for } 0 \leq \theta \leq \pi$$

$$= 2 \int_0^{\pi} \cos\frac{\theta}{2} \; dt$$

$$= 4 \sin\frac{\theta}{2} \Big]_0^{\pi} = 4 \sin\frac{\pi}{2} - 0 = 4$$

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