*Ch10: Infinite Sequences and Series.

- Sec10.1: Sequences المتتاليات

• A sequence is a list of numbers.

a_n: is the nth term. الحدود النوني

• Convergence and divergence:

The sequence $\{a_n\}$ converges to L, if $\lim_{n\to\infty} a_n = L$ where L is a finite number. Otherwise, it diverges, that is if $\lim_{n\to\infty} a_n = \mp\infty$, DNE.

So if we want to Know if the sequence converges or diverges, Just we find a limit of a_n as $n \longrightarrow \infty$. نجد النهاية حتى نحدد converge أم لا.

***Recursive definition:**

The sequence $\{a_n\}$ is defined recursively by given the values of the initial term(s) and a rule «recursion formula».

***Definitions:**

① The sequence is bounded above if there is M «finite number» such that $a_n \leq M$ for all n. محدودة من أعلى

② The sequence is bounded below if there is m «finite number» such that $a_n \geq m$ for all n.

③ The sequence is bounded if it is bounded from above and below. محدودة

④ The sequence is monotonic if it's nonincreasing غير متزايدة or nondecreasing. غير متناقصة

*** Theorem:** If the sequence is bounded and monotonic then it converges.

• Exercises: page 559

3 Find the values of a_1, a_2, a_3 and a_4:

$$a_n = \frac{(-1)^{n+1}}{2n-1}$$

$$\longrightarrow a_1 = \frac{(-1)^2}{2(1)-1} = 1$$

$$a_2 = \frac{(-1)^3}{2(2)-1} = \frac{-1}{3}$$

$$a_3 = \frac{(-1)^4}{2(3)-1} = \frac{-1}{5}.$$

$$a_4 = \frac{(-1)^5}{2(4)-1} = \frac{-1}{7}$$

11 Write out the first ten terms of the sequence.

$a_1 = a_2 = 1$, $a_{n+2} = a_{n+1} + a_n$.

$\longrightarrow$ $a_1 = 1$

$a_2 = 1$

$a_3 = a_2 + a_1 = 2$

$a_4 = a_3 + a_2 = 3$

$a_5 = a_4 + a_3 = 5$

$a_6 = a_5 + a_4 = 8$ « Fibonacci Sequence »

$a_7 = a_6 + a_5 = 13$

$a_8 = a_7 + a_6 = 21$

$a_9 = a_8 + a_7 = 34$

$a_{10} = a_9 + a_8 = 55$

16 Find a formula for the nth term of the sequence:

$1, \frac{-1}{4}, \frac{1}{9}, \frac{-1}{16}, \frac{1}{25}, \ldots$

$\longrightarrow$ $a_1 = 1$, $a_2 = \frac{-1}{4} = \frac{-1}{2^2}$, $a_3 = \frac{1}{9} = \frac{1}{3^2}, \ldots$

$\therefore a_n = \frac{(-1)^{n+1}}{n^2}$; $n = 1, 2, \ldots$. square numbers

22 The sequence $2, 6, 10, 14, 18, \dots$

$$\longrightarrow a_1 = 2 = 2(1)$$
$$a_2 = 6 = 2(3)$$
$$a_3 = 10 = 2(5)$$
$$a_4 = 14 = 2(7)$$
$$\vdots$$

$\therefore \; a_n = 2(2n-1) \;;\; n = 1, 2, 3, \dots$

* Which of the sequence converge, and which diverge:

30 $a_n = \dfrac{2n+1}{1-3\sqrt{n}}$

$$\longrightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n+1}{1-3\sqrt{n}} = -\infty$$

$\therefore \{a_n\}$ diverges.

33 $a_n = \dfrac{n^2-2n+1}{n-1}$

$$\longrightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2-2n+1}{n-1} = \infty$$

$\therefore \{a_n\}$ diverges.

$\boxed{36}$ $a_n = (-1)^n\left(1 - \frac{1}{n}\right)$.

$$\longrightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty}\left[(-1)^n\left(1-\frac{1}{n}\right)\right]$$

$$= \text{DNE}$$

$\therefore \{a_n\}$ diverges

توضيح

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(-1)^n\left(1-\frac{1}{n}\right)$$

$$= \text{DNE}.(1-0)$$

$$= \text{DNE}$$

$\boxed{38}$ $a_n = \left(2 - \frac{1}{2^n}\right)\left(3 + \frac{1}{2^n}\right)$

$$\longrightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty}\left[\left(2-\frac{1}{2^n}\right)\left(3+\frac{1}{2^n}\right)\right]$$

$$= \lim_{n\to\infty}\left(2-\frac{1}{2^n}\right)\lim_{n\to\infty}\left(3+\frac{1}{2^n}\right)$$

$$= (2)(3) = 6$$

$\therefore \{a_n\}$ converges to L.

$\boxed{42}$ $a_n = \frac{1}{(0.9)^n}$

$$\longrightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{(0.9)^n}$$

$$= \lim_{n\to\infty}\left(\frac{10}{9}\right)^n = \infty$$

$\therefore \{a_n\}$ diverges.

Note:-

$\lim_{n\to\infty} a^n = \infty$ if $a>1$

$\lim_{n\to\infty} a^n = 0$ if $0<a<1$

$\boxed{45}$ $a_n = \frac{\sin n}{n}$

$$-1 \leq \sin n \leq 1$$

$$\underbrace{\frac{-1}{n}} \leq \frac{\sin n}{n} \leq \underbrace{\frac{1}{n}}$$

$$\lim_{n\to\infty} \frac{-1}{n} = 0 \qquad \lim_{n\to\infty} \frac{1}{n} = 0$$

$\longrightarrow \lim_{n\to\infty} \frac{\sin n}{n} = 0$ by Sandwich theorem.

$\therefore \{a_n\}$ converges to $\underline{\underline{0}}$.

$\boxed{47}$ $a_n = \frac{n}{2^n}$

$$\longrightarrow \lim_{n\to\infty} \frac{n}{2^n} = \lim_{n\to\infty} \frac{1}{2^n \ln 2} = 0$$

$\therefore \{a_n\}$ converges to 0.

$\boxed{48}$ $a_n = \frac{3^n}{n^3}$

$$\longrightarrow \lim_{n\to\infty} \frac{3^n}{n^3} = \lim_{n\to\infty} \frac{3^n \ln 3}{3n^2} = \lim_{n\to\infty} \frac{3^n \ln 3 \cdot \ln 3}{6n}$$

$$= \lim_{n\to\infty} \frac{3^n (\ln 3)^3}{6} = \infty$$

$\therefore \left\{\frac{3^n}{n^3}\right\}$ diverges.

50 $a_n = \frac{\ln n}{\ln 2n}$

$\longrightarrow \lim\limits_{n\to\infty} \frac{\ln n}{\ln 2n} = \lim\limits_{n\to\infty} \frac{1/n}{\cancel{2}/\cancel{2}n} = \frac{n}{n} = 1$

$\therefore \left\{\frac{\ln n}{\ln 2n}\right\}$ converges to 1.

54 $a_n = \left(1 - \frac{1}{n}\right)^n$

$\longrightarrow \lim\limits_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = e^{-1}$

$\therefore \{a_n\}$ converges to $\frac{1}{e}$.

Note:

Theorm 5:

5 $\lim\limits_{n\to\infty} \left(1 + \frac{x}{n}\right) = e^x$; $\forall x$

59 $a_n = \frac{\ln n}{n^{1/n}}$

$\longrightarrow \lim\limits_{n\to\infty} \frac{\ln n}{n^{1/n}} = \frac{\lim_{n\to\infty} \ln n}{\lim_{n\to\infty} n^{1/n}} = \frac{\infty}{1} = \infty$

$\therefore \left\{\frac{\ln n}{n^{1/n}}\right\}$ diverges.

Note:

Theorem 5:

2 $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

$n^{\frac{1}{n}}$

62 $a_n = \sqrt[n]{3^{2n+1}}$

$$\longrightarrow \lim_{n\to\infty} \sqrt[n]{3^{2n+1}} = \lim_{n\to\infty} \left(3^{2n+1}\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} 3^{\frac{2n+1}{n}}$$

$$= \lim_{n\to\infty} 3^{2+\frac{1}{n}} = 3^2 = 9.$$

$\therefore$ $\{a_n\}$ converges to 9.

63 $a_n = \frac{n!}{n^n}$.

$$\longrightarrow \lim_{n\to\infty} \frac{n!}{n^n} = \lim_{n\to\infty} \frac{n.(n-1).(n-2).\dots(2)(1)}{n.\ n\ .\ n\ .\ \dots\ n\ .\ n}$$

$$= \lim_{n\to\infty} \frac{n}{n} . \lim_{n\to\infty} \frac{n-1}{n} . \dots . \lim_{n\to\infty} \frac{2}{n} . \lim_{n\to\infty} \frac{1}{n}$$

$$= 1 . \ 1 . \ \dots \ 0 . \ 0 = 0$$

$\therefore$ $\{a_n\}$ converges to 0.

66 $a_n = \frac{n!}{2^n . 3^n}$

$$\longrightarrow \lim_{n\to\infty} \frac{n!}{6^n} = \infty$$

$\therefore$ $\{a_n\}$ diverges.

Note:

$\lim_{n\to\infty} \frac{x^n}{n!} = 0$

$\forall x$

$\therefore \lim_{n\to\infty} \frac{n!}{x^n} = 0$

$\boxed{69}$ $a_n = \left(\frac{3n+1}{3n-1}\right)^n$

$\longrightarrow \lim_{n\to\infty}\left(\frac{3n+1}{3n-1}\right)^n \qquad 1^\infty$

We find:

$\lim_{n\to\infty} n \ln\left(\frac{3n+1}{3n-1}\right)$

$= \lim_{n\to\infty} \frac{\ln\left(\frac{3n+1}{3n-1}\right)}{1/n} \qquad «\frac{0}{0}»$

$= \lim_{n\to\infty} \frac{\frac{(3n-1)(3)-(3n+1)(3)}{(3n-1)^2}}{-1/n^2}$

$= \lim_{n\to\infty}\left(\frac{6n^2}{9n^2-6n+1}\right) = \frac{6}{9} = \frac{2}{3}$

$\therefore \lim_{n\to\infty}\left(\frac{3n+1}{3n-1}\right)^n = e^{2/3}$

So $\{a_n\}$ converges to $e^{2/3}$.

يمكن حلها أيضاً باستخدام $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x \; ; \forall x$

$\boxed{72}$ $a_n = \left(1-\frac{1}{n^2}\right)^n$

$\longrightarrow \lim_{n\to\infty}\left(1-\frac{1}{n^2}\right)^n \quad «1^\infty»$

We want to find: $\lim_{n\to\infty} n\ln\left(1-\frac{1}{n^2}\right) = \lim_{n\to\infty} \frac{\ln\left(1-\frac{1}{n^2}\right)}{1/n}$

$«\frac{0}{0}»$

$$= \lim_{n\to\infty} \frac{(2/n^3)/(1-\frac{1}{n^2})}{-1/n^2}$$

$$= \lim_{n\to\infty} \frac{-2n^2}{n^3-n} = 0$$

$$\therefore \lim_{n\to\infty}\left(1-\frac{1}{n^2}\right)^n = e^0 = 1$$

So $\{a_n\}$ converges to 1.

$\boxed{76}$ $a_n = \sinh(\ln n)$

$$\rightarrow \lim_{n\to\infty} \sinh(\ln n) = \lim_{n\to\infty} \frac{e^{\ln n} - e^{-\ln n}}{2}$$

$$= \lim_{n\to\infty} \frac{n - \frac{1}{n}}{2}$$

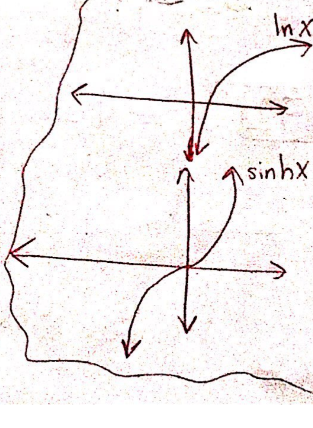
$$= \lim \frac{n^2-1}{2n} = \infty$$

ON:

$\ln n \to \infty$

$\sinh n \to \infty$

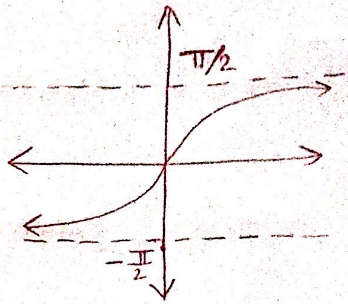
$\therefore \sinh(\ln n) \to \infty$



$\therefore$ $\{a_n\}$ diverges.

$\boxed{81}$ $a_n = \tan^{-1} n$

$$\longrightarrow \lim_{n\to\infty} \tan^{-1} n = \frac{\pi}{2}$$



$\therefore \{a_n\}$ converges to $\frac{\pi}{2}$.

$\boxed{82}$ $a_n = \frac{1}{\sqrt{n}} \tan^{-1} n$

$$\longrightarrow \lim_{n\to\infty} \frac{\tan^{-1} n}{\sqrt{n}} = 0$$

$\therefore \{a_n\}$ converges to 0.

Note:

$\tan^{-1} n \longrightarrow \frac{\pi}{2}$

$\sqrt{n} \longrightarrow \infty$

$\therefore \frac{\tan^{-1} n}{\sqrt{n}} \longrightarrow 0$

$\boxed{89}$ $a_n = \frac{1}{n} \int_1^n \frac{1}{x} dx$.

$$\longrightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} \int_1^n \frac{1}{x} dx$$

$$= \lim_{n\to\infty} \frac{1}{n} \left(\ln x \Big|_1^n \right)$$

$$= \lim_{n\to\infty} \frac{\ln n}{n} = 0$$

$\therefore \{a_n\}$ converges to 0

*Assume that each sequence converges and find its limit.

$\boxed{91}$ $a_1 = 2$, $a_{n+1} = \frac{72}{1+a_n}$

$\longrightarrow \{a_n\}$ converges so $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1} = M$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{72}{1+a_n}$$

$$M = \frac{72}{1+M}$$

$$M^2 + M - 72 = 0$$

$$(M+9)(M-8) = 0$$

$$M = \cancel{-9} \text{ or } 8$$

but $a_1 = 2$ and $a_{n+1} = \frac{72}{1+a_n}$ so a_n is +ve $\forall n$

$\therefore \lim\limits_{n\to\infty} a_n = 8$

$\boxed{92}$ $a_1 = -1$, $a_{n+1} = \dfrac{a_n + 6}{a_n + 2}$

$\longrightarrow \{a_n\}$ converges, so:

$$\lim a_{n+1} = \lim_{n\to\infty} \frac{a_n + 6}{a_n + 2}$$

$$L = \frac{L+6}{L+2}$$

$$L^2 + 2L = L + 6$$

$$L^2 + L - 6 = 0$$

$$(L+3)(L-2) = 0$$

$$\therefore L = \cancel{-3}, 2$$

$a_1 = -1$, $a_2 = \frac{5}{1}$, $a_3 = \frac{11}{7}$, ... so a_n +ve $\forall n > 1$

$$\therefore \lim_{n\to\infty} a_n = 2$$

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*Determine if the sequence is monotonic and if it is bounded:

111 $a_n = \dfrac{3n+1}{n+1}$

Note: monotonicity

نستطيع أيضاً من خلال اثبات أن
$a_{n+1} > a_n$ or
$a_{n+1} < a_n$

$\rightarrow f(x) = \dfrac{3x+1}{x+1}$

$f'(x) = \dfrac{(x+1)(3) - (3x+1)(1)}{(x+1)^2}$

$= \dfrac{2}{(x+1)^2}$ +ve

∴ $f(x)$ is increasing $\longrightarrow$ $\{a_n\}$ is nondecreasin so it's monotonic.

$\rightarrow 2, \frac{7}{3}, \text{----}, 3$ (increasing)

so $2 \leqslant a_n \leqslant 3$ ∴ $\{a_n\}$ is bounded.

114 $a_n = 2 - \dfrac{2}{n} - \dfrac{1}{2^n}$

$\longrightarrow f(x) = 2 - \dfrac{2}{x} - \dfrac{1}{2^x}$

$f'(x) = \dfrac{2}{x^2} + \dfrac{1}{2^x} \cdot \ln 2$ +ve

$f(x)$ is increasing $\longrightarrow$ $\{a_n\}$ is monotonic.

$\xrightarrow{}\ \frac{-1}{2}, \frac{1}{2}, \ldots, 2$ (increasing)

so $\frac{-1}{2} \leq a_n \leq 2$ $\quad \therefore \{a_n\}$ is bounded.

$\boxed{127}$ Is it true that a sequence $\{a_n\}$ of positive numbers must converge if it's bounded from above?

No, Counterexample: « مثال يعكس الجملة أي لا يحققها »

$a_n = 3 + (-1)^n$

$\rightarrow 2, 4, 2, 4, \ldots$

a_n is positive for all n, and

$a_n \leq 4$ so it's bounded above

but $\lim\limits_{n\to\infty} a_n = DNE$, so $\{a_n\}$ diverges.