Ch10: Infinite Sequences and Series

10.1: Sequences

Representing Sequences

A sequence is a list of numbers

$$a_1, a_2, a_3, a_4, \dots, a_n, \dots$$

Example:

$2, 4, 6, 8, 10, 12, \dots, 2n$

$a_n = 2n, \; n = 1, 2, 3, \dots$

Example: $\underbrace{12}_{10+2}, \underbrace{14}_{10+2\cdot 2}, 16, 18, 20, 22, \dots$

$a_n = 10 + 2n, \; n = 1, 2, \dots$

Example:

$a_n = \sqrt{n}$ $\qquad$ $b_n = (-1)^{n+1} \cdot \frac{1}{n}$

$a_1 = 1$ $\qquad$ $b_1 = 1$

$a_2 = \sqrt{2}$ $\qquad$ $b_2 = -\frac{1}{2}$

$a_3 = \sqrt{3}$ $\qquad$ $b_3 = \frac{1}{3}$

$a_4 = \sqrt{4}$ $\qquad$ $b_4 = -\frac{1}{4}$

$\{b_n\} = \left\{(-1)^{n+1} \frac{1}{n}\right\} = \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots\right\}$

$\{a_n\} = \{\sqrt{n}\}_{n=1}^{\infty} = \{1, \sqrt{2}, \sqrt{3}, \sqrt{4}, \dots\}$

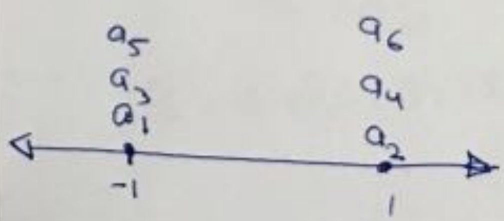
For sequences with nth term a_n

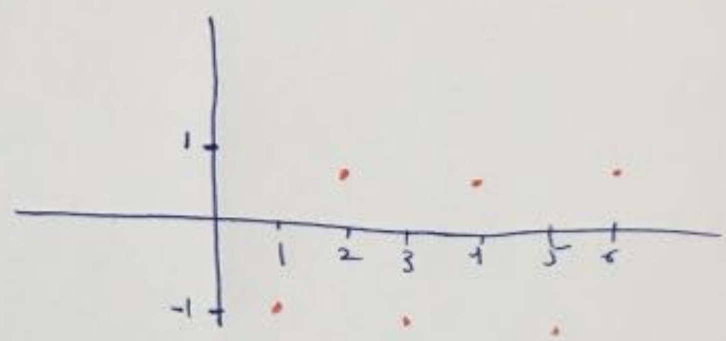
$$\{a_n\}_{n=1}^{\infty}.$$

We can consider the sequence $\{a_n\}$ as a function whose domain is the integer values and its range is $a_n, n=1,2,\ldots$

So we have $(1,a_1),(2,a_2),(3,a_3),\ldots$

Example: Represent $a_n=(-1)^n$





Convergence and Divergence.

Sometimes the numbers in a sequence approach a single value as the index $\boxed{n}$ increases

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots, \frac{1}{n}, \dots\right\} \longrightarrow 0$$

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$$\left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, 1-\frac{1}{n}, \dots\right\} \longrightarrow 1-0=1$$

$$\left\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \dots, \sqrt{n}\right\} \text{ Diverges}$$

$$\left\{1, -1, 1, -1, 1, -1, \dots, (-1)^{n+1}, \dots\right\} \text{ Diverge.}$$

Definition:

The sequence $\{a_n\}$ converges to the number L if for every positive number ϵ there correspond an integer N; for all n,

$$n > N \longrightarrow |a_n - L| < \epsilon$$

If no such L exists, we say that $\{a_n\}$ diverges

If $\{a_n\}$ converge to L, we write, $\lim_{n\to\infty} a_n = L$ or simply $a_n \to L$. and call L the limit of the sequence

Definition:

The sequence $\{a_n\}$ diverges to ∞ if for $\forall$ M there is an integer N; $\forall$ n larger that N, $a_n > M$. If this condition holds we write

$$\lim_{n\to\infty} a_n = \infty \quad \text{or} \quad a_n \to \infty$$

Similarly if for every number m there is an integer N such that for all $n > N$ we have $a_n < m$, then we say $\{a_n\}$ diverge negatively to $-\infty$

$$\lim_{n\to\infty} a_n = -\infty \quad \text{or} \quad a_n \to -\infty$$

Calculating Limits of sequences

Theorem 1: let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$

1. $\lim_{n\to\infty} (a_n + b_n) = A + B$
2. $\lim_{n\to\infty} (a_n - b_n) = A - B$
3. $\lim_{n\to\infty} k \cdot b_n = k \cdot B$
4. $\lim_{n\to\infty} (a_n \cdot b_n) = A \cdot B$
5. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$, if $B \neq 0$

Example:

ⓐ $\lim \frac{-1}{n} = -1 \cdot \lim_{n\to\infty} \frac{1}{n} = -1.0 = 0$

ⓑ $\lim_{n\to\infty} \left(\frac{n-1}{n}\right) = \lim_{n\to\infty}\left(1-\frac{1}{n}\right) = \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{n} = 1-0 = 1$

ⓒ $\lim_{n\to\infty} \frac{5}{n^2} = 5 \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty} \frac{1}{n} = 5 \cdot 0 \cdot 0 = 5$

ⓓ $\lim_{n\to\infty} \frac{4-7n^6}{n^6+3} = \lim_{n\to\infty} \frac{\frac{4}{n^6}-7}{1+\frac{3}{n^6}} = \frac{-7}{1} = -7$

Example:

$\{a_n\} = \{1, 2, 3, 4, \ldots\}$ Diverges

$\{b_n\} = \{-1, -2, -3, \ldots,\}$ Diverges

$\{a_n + b_n\} = \{0, 0, 0, 0, \ldots,\}$ Converge

any nonzero multiple of a divergent sequence $\{a_n\}$ diverges

Theorem- The Sandwich Th. for sequences

let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers

If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N

and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ then $\lim_{n\to\infty} b_n = L$ also

Example:

① $\frac{\cos n}{n}$

$$\underset{\substack{\downarrow \\ 0}}{\frac{-1}{n}} \le \underset{\substack{\text{S.T} \Downarrow \\ 0}}{\frac{\cos n}{n}} \le \underset{\substack{\downarrow \\ 0}}{\frac{1}{n}}$$

② $\frac{1}{2^n}$

$$\underset{\substack{\downarrow \\ 0}}{0} \le \underset{\substack{\text{S.T} \Downarrow \\ 0}}{\frac{1}{2^n}} \le \underset{\substack{\downarrow \\ 0}}{\frac{1}{n}}$$

③ $(-1)^n \cdot \frac{1}{n}$

$$\underset{\substack{\downarrow \\ 0}}{-\frac{1}{n}} \le \underset{\substack{\text{S.T} \Downarrow \\ 0}}{(-1)^n \cdot \frac{1}{n}} \le \underset{\substack{\downarrow \\ 0}}{\frac{1}{n}}$$

Theorem: The Continuous function Th. for sequences

Let $\{a_n\}$ be a sequence of real numbers

If $\boxed{a_n \to L}$ and if $\boxed{f}$ is a function that is continuous at L and defined at all a_n, then $\boxed{f(a_n) \to f(L)}$

Example:

Show that $\sqrt{\frac{n+1}{n}} \to 1$

$\frac{n+1}{n} \longrightarrow 1$

$\sqrt{\frac{n+1}{n}} \to \sqrt{1} = 1$

$a_n = \frac{n+1}{n}$ $\quad a_n \to 1$

$f(x) = \sqrt{x}$ $\quad f(a_n) \to f(1)$

$\sqrt{a_n} \to \sqrt{1} = 1$

Example: $\left\{\frac{1}{n}\right\} \longrightarrow 0$

$2^{\frac{1}{n}} \longrightarrow 2^0 = 1$

So $a_n = \frac{1}{n}$ $\quad a_n \longrightarrow 0$ $\quad \cdot L = 0$

$f(x) = 2^x$

$2^{\frac{1}{n}} = f\left(\frac{1}{n}\right) \to f(L) = 2^0 = 1$

Using L'Hôpital's Rule

Theorem: Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ $\forall n \geq n_0$, then.

$$\boxed{\lim_{x \to \infty} f(x) \Longrightarrow \lim_{n \to \infty} a_n = L}$$

Example: $\lim_{n \to \infty} \frac{\ln n}{n}$

$$\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0$$

Example: Does the sequence $\{a_n\} = \left\{\left(\frac{n+1}{n-1}\right)^n\right\}$ converge?

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \left(\frac{n+1}{n-1}\right)^n \qquad (1^\infty)$$

$$\lim_{n \to \infty} \ln a_n = \lim_{n \to \infty} n \ln\left(\frac{n+1}{n-1}\right) = \lim_{n \to \infty} \left[\frac{\ln\left(\frac{n+1}{n-1}\right)}{\frac{1}{n}}\right] = \lim_{n \to \infty} \frac{\frac{-2}{n^2-1}}{\frac{-1}{n^2}}$$

$$= \lim_{n \to \infty} \frac{2n^2}{n^2-1} = 2$$

So $\lim_{n \to \infty} a_n = e^2$

Commonly Occurring Limits

1) $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

2) $\lim_{n\to\infty} \sqrt[n]{n} = 1$

3) $\lim_{n\to\infty} x^{\frac{1}{n}} = 1$

4) $\lim_{n\to\infty} x^n = 0 \ (|x| < 1)$

5) $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$ (any x)

6) $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ (any x)

In Formulas (3) through (6) x remains fixed as $n\to\infty$

Example:

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a) $\lim_{n\to\infty} \frac{\ln n^2}{n} = \lim_{n\to\infty} \frac{2\ln n}{n} = 2.0 = 0$

b) $\lim_{n\to\infty} \sqrt[n]{n^2} = \lim_{n\to\infty} \left(\sqrt[n]{n}\right)^2 = (1)^2 = 1$

c) $\sqrt[n]{3n} = (3n)^{\frac{1}{n}} = 3^{\frac{1}{n}} \cdot n^{\frac{1}{n}} \rightarrow 1.1 = 1$

d) $\left(\frac{-1}{2}\right)^n \longrightarrow 0$

e) $\left(\frac{n-2}{n}\right)^n = \left(1-\frac{2}{n}\right)^n \longrightarrow e^{-2}$

f) $\frac{100^n}{n!} \to 0$

Recursive Definitons:

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Example:

① $a_n = a_{n-1} + 1 \,,\; a_1 = 1$

$a_1 = 1$

$a_2 = a_1 + 1 = 1 + 1 = 2$

$a_3 = a_2 + 1 = 2 + 1 = 3$

$a_4 = a_3 + 1 = 3 + 1 = 4$

$\vdots$

② $a_n = n\, a_{n-1} \,,\; n > 1 \,,\; a_1 = 1$

$a_1 = 1$

$a_2 = 2 \cdot a_1 = 2 \cdot 1 = 2$

$a_3 = 3\, a_2 = 3 \cdot 2 = 6$

$a_4 = 4\, a_3 = 4 \cdot 6 = 24$

$\vdots$

③ $a_{n+1} = a_n + a_{n-1} \,,\; n > 2$

$a_1 = 1, \; a_2 = 1$

$a_3 = a_2 + a_1 = 1 + 1 = 2$

$a_4 = a_3 + a_2 = 2 + 1 = 3$

$a_5 = 5 \ldots$

(46) $\{a_n\} = \left\{\dfrac{\sin^2 n}{2^n}\right\}$

Does the sequence $\{a_n\}$ converge?

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sin^2 n}{2^n}$$

$$0 \le \underline{\sin^2 n} \le 1$$

$$0 \le \frac{\sin^2 n}{2^n} \le \frac{1}{2^n}$$

$$\begin{array}{ccccc} 0 & \le & \dfrac{\sin^2 n}{2^n} & \le & \dfrac{1}{2^n} \\ \downarrow & & \Downarrow \text{S.T} & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

So the sequence $\{a_n\}$ converges

(59) $a_n = \dfrac{\ln n}{n^{\frac{1}{n}}} = \infty$ "Diverges"

(68) $a_n = \ln\left(1+\frac{1}{n}\right)^n$

$$\lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n = \ln\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right) = \ln\left(e^1\right) = 1$$

So $\{a_n\}$ converges

84 $a_n = \sqrt[n]{n^2+n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt[n]{n^2+n}$$

$$= \lim_{n\to\infty} \left(n^2+n\right)^{\frac{1}{n}}$$

$$\lim_{n\to\infty} \ln\left(n^2+n\right)^{\frac{1}{n}} = \lim_{n\to\infty}\left[\frac{\ln(n^2+n)}{n}\right] = \lim_{n\to\infty}\left[\frac{2n+1}{(n^2+n)}\right] = 0$$

so $\lim_{n\to\infty} \sqrt[n]{n^2+n} = e^0 = 1$

Bounded Monotonic Sequences

Definition:

A sequence $\{a_n\}$ is bounded from above if there exists a number M such that $a_n \le M \ \forall n$.

The number M is $\boxed{\text{an upper bound}}$ for $\{a_n\}$

If M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the $\boxed{\text{least upper b.}}$ bound for $\{a_n\}$

A sequence $\{a_n\}$ is bounded from below, if there exists a number m such that $a_n \ge m \ \forall n$. The number m is a $\boxed{\text{Lower bound for } \{a_n\}}$

If m is a Lower bound for $\{a_n\}$ but no number greater that m is a Lower bound for $\{a_n\}$, then m is the $\boxed{\text{greatest Lower bound for}}$ $\{a_n\}$. ~~the~~

If $\{a_n\}$ is bounded from above and below, then $\{a_n\}$ is bounded.

If $\{a_n\}$ is not bounded (bdd) then $\{a_n\}$ is said to be unbounded.

Example:

① The sequence $1, 2, 3, 4, \dots, n, \dots$
has no Lower bound
It's bounded below by any number less than or equal to 1

G.L.B = greatest Lower bound is 1

② $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots, \frac{n}{n+1}, \dots$
is bdd above by any number greater than or equal to 1

$M = 1$ Least upper bound
$m = \frac{1}{2}$: greatest Lower bound

Definition:

A sequence $\{a_n\}$ is $\boxed{\text{nondecreasing}}$ if $a_n \le a_{n+1}$ $\forall n$
that is $a_1 \le a_2 \le a_3 \le a_4 \le \dots$
The sequence is $\boxed{\text{nonincreasing}}$ if $a_n \geq a_{n+1}$ $\forall n$.
The sequence is $\boxed{\text{monotonic}}$ if it either nonincreasing or nondecreasing

Example:

① $1, 2, 3, 4, \dots, n, \dots$ nondecreasing
② $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{n}{n+1}, \dots$ nondecreasing
③ $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots, \frac{1}{2^n}, \dots$: nonincreasing

④ 3, 3, 3, …, 3, …. both nondecreasing and non-increasing

⑤ 1, -1, 1, -1, 1, -1, … not monotonic

Theorem: The monotonic sequence th.

If a sequence $\{a_n\}$ is both bdd and monotonic, then the sequence converges