a

Combining Series

Theorem 8: If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then.

1. Sum Rule: $\sum(a_n + b_n) = \sum a_n + \sum b_n = A+B$
2. Difference Rule: $\sum(a_n - b_n) = \sum a_n - \sum b_n = A-B$
3. Constant Multiple Rule: $\sum k a_n = k\sum a_n = kA$ " any #k "

Proof:

$A_n = a_1 + a_2 + a_3 + \cdots + a_n$ (nth partial sum of $\sum a_n$)

$B_n = b_1 + b_2 + \cdots + b_n$. (nth partial sum of $\sum b_n$)

S_n of $\sum(a_n + b_n)$

$$S_n = (a_1+b_1) + (a_2+b_2) + (a_3+b_3) + \cdots + (a_n+b_n)$$

$$= (a_1 + a_2 + a_3 + \cdots + a_n) + (b_1 + b_2 + \cdots + b_n)$$

$$= A_n + B_n$$

Since $A_n \to A$ & $B_n \to B$

So. $S_n \to A+B$

Do 2 and 3

Remarks

① Every nonzero constant multiple of a divergent series diverge

② If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum (a_n + b_n)$ and $\sum (a_n - b_n)$ both diverge.

~~Cation~~
Caution: Remember $\sum a_n + b_n$ can converge when $\sum a_n$ and $\sum b_n$ both diverge.

$\sum a_n = 1+1+1+\cdots+1$ Diverges

$\sum b_n = -1 + -1 + -1 + \cdots + -1$ Diverge

$\sum (a_n + b_n) = 0+0+\cdots+0$ converge to 0

Example:

a) $\sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}} = \sum_{n=1}^{\infty} \left(\left(\frac{3}{6}\right)^{n-1} - \frac{1}{6^{n-1}} \right)$, both converge

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}}$$

$$= \frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{1}{6}} = 2 - \frac{6}{5} = \frac{4}{5}$$

c

(b) $\sum_{n=0}^{\infty} \frac{4}{2^n} = 4\sum_{n=0}^{\infty} \frac{1}{2^n}$

$$= 4\cdot\left(\frac{1}{1-\frac{1}{2}}\right) = 8$$

Adding or Deleting Terms

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_{k-1} + \sum_{n=k}^{\infty} a_n$$

If $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=k}^{\infty} a_n$ converges for $k>1$

Conversely,

If $\sum_{n=k}^{\infty} a_n$ converges for any $k>1$, then $\sum_{n=1}^{\infty} a_n$ conv.

Thus.

$$\sum_{n=1}^{\infty} \frac{1}{5^n} = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \sum_{n=4}^{\infty} \frac{1}{5^n}$$

and $\sum_{n=4}^{\infty} \frac{1}{5^n} = \left(\sum_{n=1}^{\infty} \frac{1}{5^n}\right) - \frac{1}{5} - \frac{1}{25} - \frac{1}{25}$

Re indexing

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1+h}^{\infty} a_{n-h} = a_1 + a_2 + a_3 + \dots$$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1-h}^{\infty} a_{n+h} = a_1 + a_2 + \dots$$

Example:

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \dots$$

$$\sum_{n=0}^{\infty} \frac{1}{2^{n}} = \sum_{n=5}^{\infty} \frac{1}{2^{n-5}} = \sum_{n=-4}^{\infty} \frac{1}{2^{n+4}}$$

The partial sums remain the same