- Sec 10.3: Infinite Series.

* An infinite series is the sum of an infinite sequence.

$$\text{“ } \sum_{n=1}^{\infty} a_n \text{ ”}$$

* Convergence and divergence:

① n-th partial sum “S_n”

$$S_n = a_1 + a_2 + \dots + a_n$$

إذا استطعنا إيجاد صيغة لـ S_n وأوجدنا
limit وكانت :

→ $\lim_{n\to\infty} S_n = L$; where L is a finite number.

then the series converges to L, that is $\sum_{n=1}^{\infty} a_n = L$.

→ $\lim_{n\to\infty} S_n = \infty, -\infty$ or DNE, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

• Note: This test is usually used if we have a telescoping series.

② Geometric series:

is the sum of an infinite number of

terms that have a constant ratio between successive terms.

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots \, .$$

If $|r| < 1$ « that is, $-1 < r < 1$ », then the geometric series converges to

$$\frac{a}{1-r} \quad ; \quad \begin{array}{l} a: \text{first term} \\ r: \text{ratio} \end{array}$$

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If $|r| \geq 1$, then it diverges.

3 n-th term test: (for divergent series).

If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges.

If $\lim_{n\to\infty} a_n = 0$, the the series may converge or diverge.

(أي أنه إذا كانت الحدود تذهب إلى الصفر لا نستطيع الحكم على series).

But if the series $\sum a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

* Exercises: page 569.

6 Find a formula for the n-th partial sum and use it to find the series' sum:-

$$\frac{5}{1(2)}+\frac{5}{2(3)}+\cdots+\frac{5}{n(n+1)}+\cdots$$

telescoping series.

$$\longrightarrow a_n=\frac{5}{n(n+1)}=\frac{A}{n}+\frac{B}{n+1}$$

$$5=A(n+1)+Bn$$

$n=0$: $A=5$

$n=-1$: $B=-5$

$$\therefore \sum_{n=1}^{\infty} a_n=\sum_{n=1}^{\infty}\left(\frac{5}{n}-\frac{5}{n+1}\right)$$

the n-the partial sum:

$$S_n=a_1+a_2+\cdots+a_n$$

$$=\left(5-\cancel{\frac{5}{2}}\right)+\left(\cancel{\frac{5}{2}}-\cancel{\frac{5}{3}}\right)+\left(\cancel{\frac{5}{3}}-\cancel{\frac{5}{4}}\right)+\cdots$$

$$+\left(\cancel{\frac{5}{n}}-\frac{5}{n+1}\right)$$

$$S_n=5-\frac{5}{n+1}$$

Now $\lim_{n\to\infty} S_n=\lim_{n\to\infty}\left(5-\frac{5}{n+1}\right)=5$

$\therefore \sum_{n=1}^{\infty}\frac{5}{n(n+1)}$ converges to 5, that

is $\sum_{n=1}^{\infty}\frac{5}{n(n+1)}=5$.

$\boxed{14}$ $\displaystyle\sum_{n=0}^{\infty} \frac{2^{n+1}}{5^n}$ «Geometric series».

$$\longrightarrow \sum_{n=0}^{\infty} \frac{2^{n+1}}{5^n} = 2 + \frac{4}{5} + \frac{8}{25} + \cdots$$

$$a = 2, \quad r = \frac{2}{5} \quad «-1 < r < 1»$$

$$\therefore \sum_{n=1}^{\infty} \frac{2^{n+1}}{5^n} \text{ converges to } \frac{a}{1-r} = \frac{2}{1-(2/5)} = \frac{10}{3}$$

$$«\text{that is, } \sum_{n=0}^{\infty} \frac{2^{n+1}}{5^n} = \frac{10}{3}$$

$\boxed{18}$ $\left(\frac{-2}{3}\right)^2 + \left(\frac{-2}{3}\right)^3 + \left(\frac{-2}{3}\right)^4 + \cdots$

Geometric series

$$a = \left(\frac{-2}{3}\right)^2, \quad r = \frac{-2}{3} \quad «-1 < r < 1»$$

$$\therefore \sum_{n=2}^{\infty} \left(\frac{-2}{3}\right)^n \text{ converges to } \frac{a}{1-r} = \frac{(4/9)}{1-(-2/3)} = \frac{4}{15}$$

$$«\text{that is, } \sum_{n=2}^{\infty} \left(\frac{-2}{3}\right)^n = \frac{4}{15}».$$

$\boxed{24}$ Write the number as a fraction $1.\overline{414}$

$\rightarrow 1.\overline{414} = 1.414\,414\ldots$

$= 1 + \underline{0.414 + 0.000414 + \ldots}$

geometric series
$a = 0.414,\ r = 0.001$
$\therefore$ converges to $\dfrac{0.414}{1-0.001}$

$= \dfrac{0.414}{0.999} = \dfrac{414}{999}$

$\therefore 1.\overline{414} = 1 + \dfrac{414}{999} = \dfrac{1413}{999}.$

$\boxed{3\ 2}$ Which series converge, and which diverge:

$$\sum_{n=0}^{\infty} \frac{e^n}{e^n+n}$$

by nth term test:

$\rightarrow \lim_{n\to\infty} \dfrac{e^n}{e^n+n}$ $\left(\dfrac{\infty}{\infty}\right)$ L'Hopital's rule

$= \lim_{n\to\infty} \dfrac{e^n}{e^n+1} \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \dfrac{e^n}{e^n} = 1 \neq 0$

$\therefore \sum_{n=0}^{\infty} \dfrac{e^n}{e^n+n}$ diverges by n-th term test

$\boxed{38}$ $\sum_{n=1}^{\infty}(\tan n - \tan(n-1))$

« telescoping series »

$\longrightarrow S_n = a_1 + a_2 + a_3 + \cdots + a_n$

$= (\cancel{\tan 1} - \tan 0) + (\cancel{\tan 2} - \cancel{\tan 1})$

$+ (\cancel{\tan 3} - \cancel{\tan 2}) + \cdots + (\tan n - \cancel{\tan(n-1)})$

$= -\underbrace{\cancel{\tan 0}}_{0} + \tan n = \tan n$

$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} \tan n = DNE$

So $\sum_{n=1}^{\infty}(\tan n - \tan(n-1))$ diverges by nth partial sum.

$\boxed{44}$ $\sum_{n=1}^{\infty}\frac{2n+1}{n^2(n+1)^2}$

by n-th partial sum.

$\longrightarrow \sum_{n=1}^{\infty}\frac{2n+1}{n^2(n+1)^2} = \sum_{n=1}^{\infty}\left(\frac{A}{n^2} + \frac{B}{(n+1)^2}\right)$

$2n+1 = A(n+1)^2 + Bn^2$

$\underline{2n} + \underline{1} = \underbrace{(A+B)}_{0}n^2 + \underline{2An} + \underline{A}$

$$A = 1$$
$$A + B = 0 \longrightarrow B = -1$$

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \right)$$

$$\therefore S_n = a_1 + a_2 + a_3 + \dots + a_n$$

$$= \left(\frac{1}{1} - \cancel{\frac{1}{4}} \right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{9}} \right) + \left(\cancel{\frac{1}{9}} - \cancel{\frac{1}{16}} \right) + \dots + \left(\cancel{\frac{1}{n^2}} - \frac{1}{(n+1)^2} \right).$$

$$= 1 - \frac{1}{(n+1)^2}$$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left(1 - \frac{1}{(n+1)^2} \right) = 1$$

$$\therefore \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} \text{ converges to } 1.$$

« That is, $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = 1$ ».

$\boxed{54}$ $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{5^n}$

We note that $\cos(n\pi)$: $1, -1, 1, -1, \ldots$

$\therefore \cos(n\pi) = (-1)^n$

$$\longrightarrow \sum_{n=0}^{\infty} \frac{\cos(n\pi)}{5^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{5^n}$$

$$= 1 - \frac{1}{5} + \frac{1}{25} - \cdots\cdots$$

geometric series.

$a = 1$, $r = \frac{-1}{5}$ $\ll -1 < r < 1 \gg$

$$\therefore \sum_{n=0}^{\infty} \frac{\cos(n\pi)}{5^n} \text{ converges to } \frac{a}{1-r} = \frac{1}{1-(-1/5)} = \frac{5}{6}$$

$\boxed{62}$ $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

by n-th term test

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^n}{n!} = \lim_{n\to\infty} \frac{n \cdot n \cdot n \cdot \cdots \cdot n \cdot n}{n(n-1)(n-2)\cdots(2)(1)}$$

$$= \lim_{n\to\infty} \frac{n}{n} \cdot \lim_{n\to\infty} \frac{n}{n-1} \cdot \cdots \cdot \lim n$$

$$= 1 \cdot 1 \cdot \cdots \cdot \infty$$

$$= \infty \neq 0$$

$$\therefore \sum_{n=1}^{\infty} \frac{n^n}{n!} \text{ diverges by n-th term test.}$$

$\boxed{63}$ $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n}$

$$\longrightarrow \sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n} = \sum_{n=1}^{\infty} \left(\left(\frac{2}{4}\right)^n + \left(\frac{3}{4}\right)^n \right)$$

$$= \underbrace{\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n}_{1} + \underbrace{\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n}_{2}$$

2- geometric series

1 $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

$$a = \frac{1}{2}, \quad r = \frac{1}{2} \quad (-1 < r < 1)$$

$$\therefore \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{(1/2)}{1 - (1/2)} = 1$$

2 $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n = \frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \dots$

$$a = \frac{3}{4}, \quad a = \frac{3}{4}$$

$$\therefore \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n = \frac{(3/4)}{1 - (3/4)} = 3$$

$$\longrightarrow \sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n} = 1 + 3 = 4$$

$\boxed{78}$ Find the values of x for which the series converges and find the sum $\displaystyle\sum_{n=0}^{\infty} (\ln x)^n$.

$$\longrightarrow \sum_{n=0}^{\infty} (\ln x)^n = 1 + \ln x + (\ln x)^2 + \dots .$$

geometric series.

$a = 1$, $r = \ln x$

$\sum_{n=0}^{\infty} (\ln x)^n$ converges if $-1 < r < 1$

that is, $-1 < \ln x < 1$

this implies $e^{-1} < x < e$

So if $x \in (e^{-1}, e)$, then the series $\sum_{n=0}^{\infty} (\ln x)^n$ converges to $\dfrac{1}{1 - \ln x}$.

"That is, $\sum_{n=0}^{\infty} (\ln x)^n = \dfrac{1}{1 - \ln x}$; $e^{-1} < x < e$"

$\boxed{90}$ Find the value of b for which

$$1 + e^b + e^{2b} + \dots = 9$$

geometric series

$\longrightarrow a = 1$, $r = e^b$

$\therefore 1 + e^b + e^{2b} + \dots = \dfrac{1}{1 - e^b} = 9$

$$\frac{1}{1-e^b} = 9$$

$$9 - 9e^b = 1$$
$$-9 \qquad\qquad -9$$

$$\frac{-9e^b}{-9} = \frac{-8}{-9}$$

$$e^b = \frac{8}{9}$$

$$\ln e^b = \ln\left(\frac{8}{9}\right)$$

$$b = \ln\left(\frac{8}{9}\right).$$