Telescoping Series

Example:

Find the sum of telescoping $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$$\frac{1}{n(n+1)} = \frac{a}{n} + \frac{b}{n+1}$$

$$1 = a(n+1) + bn$$

$n=0$ $\quad 1 = a$

$n=-1$ $\quad 1 = -b \rightarrow b = -1$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

$$S_k = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{k-1} - \frac{1}{k}\right) + \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$S_k = 1 - \frac{1}{k+1}$$

as $k \rightarrow \infty$

$$S_k \rightarrow 1 - 0 = 1$$

So the series converges and its sum is 1

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

Example:

41 $\displaystyle\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)}$

$$\frac{4}{(4n-3)(4n+1)} = \frac{a}{4n-3} + \frac{b}{4n+1}$$

$$4 = a(4n+1) + b(4n-3)$$

$n = \frac{-1}{4}$

$$4 = a(0) + b(-4)$$

$$b = -1$$

$n = \frac{3}{4}$

$$4 = a \cdot 4 + b(0)$$

$$a = 1$$

So $\displaystyle\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)} = \sum_{n=1}^{\infty} \frac{1}{4n-3} + \frac{-1}{4n+1}$

$$S_k = 1 - \frac{1}{5} + \frac{1}{5} - \frac{1}{9} + \frac{1}{9} - \frac{1}{13} + \cdots + \underbrace{\frac{1}{4k-7} - \frac{1}{4k-3}}_{n=k-1} + \frac{1}{4k-3} - \frac{1}{4k+1}$$

$$S_k = 1 - \frac{1}{4k+1}$$

as $k \to \infty$, $S_k \to 1 - 0 = 1$

So $\displaystyle\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)} = 1$

(51) $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{3}{2^n}$

$\frac{3}{2} - \frac{3}{2^2} + \frac{3}{2^3} - \frac{3}{2^4} + \frac{3}{2^5} - \cdots$

Geometric series

$a = \frac{3}{2}$

$r = \frac{\frac{-3}{2^2}}{\frac{3}{2}} = \frac{-1}{2}$

$|r| = \frac{1}{2} < 1$ so the series converges to

$\frac{a}{1-r} = \frac{\frac{3}{2}}{1 - -\frac{1}{2}} = \frac{\frac{3}{2}}{\frac{3}{2}} = 1$

(75) $\sum_{n=0}^{\infty} (-1)^n (x+1)^n$ Find the values of x for which the given geometric series converge. Also find its sum

$1 - (x+1) + (x+1)^2 - (x+1)^3 + \cdots$

Geometric

converges if $|x+1| < 1$, $r = -(x+1)$

$-1 < x+1 < 1$

$-2 < x < 0$

Sum $= \frac{a}{1-r} = \frac{1}{1 - -(x+1)} = \frac{1}{1+x+1} = \frac{1}{2+x}$