- Sec 10.4: Comparison Tests :- اختبارات المقارنة

• D.C.T «Direct comparison test» :-

let $\sum a_n$, $\sum b_n$, $\sum C_n$ be series with nonnegative terms , if

① $a_n \leq b_n$ and $\sum b_n$ converges then $\sum a_n$ converges.

② $C_n \leq a_n$ and $\sum C_n$ diverges then $\sum a_n$ diverges.

• L.C.T «limit comparison test» :-

let $\sum a_n$ and $\sum b_n$ be series with positive terms: $\forall n \geq N$

① If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ «any number >0» then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

② If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges then $\sum a_n$ converges

③ If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges , then $\sum a_n$ diverges.

$\sum a_n$: المراد فحصها و $\sum b_n$: التي نقارن معها

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$\boxed{8}$ Determine if the series converges or diverges:-

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{\sqrt{n^2+3}}$$

$$\longrightarrow a_n = \frac{\sqrt{n}+1}{\sqrt{n^2+3}} \quad , \quad b_n = \frac{1}{\sqrt{n}}$$

by L.C.T:-

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty}\left(\frac{\sqrt{n}+1}{\sqrt{n^2+3}} \div \frac{1}{\sqrt{n}}\right)$$

$$= \lim_{n\to\infty} \frac{n+\sqrt{n}}{\sqrt{n^2+3}}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}\left(1+\frac{1}{\sqrt{n}}\right)}{\cancel{n}\sqrt{1+\frac{3}{n^2}}} = \frac{1+0}{\sqrt{1+0}} = 1 > 0$$

$\therefore$ $\sum a_n$ and $\sum b_n$ both converge or both diverge.

but $\sum b_n = \sum \frac{1}{\sqrt{n}}$ diverges «p-series with $p=\frac{1}{2}$»

$\therefore \rightarrow \sum a_n = \sum \frac{\sqrt{n}+1}{\sqrt{n^2+3}}$ diverges by L.C.T.

*يجوز حلها بـ D.C.T :-

$$\frac{\sqrt{n}}{\sqrt{n^2+3n^2}} \leq \frac{\sqrt{n}}{\sqrt{n^2+3}} \leq \frac{\sqrt{n}+1}{\sqrt{n^2+3}}$$

$$\frac{\sqrt{n}}{\sqrt{4n^2}} = \frac{1}{2\sqrt{n}} \leq \frac{\sqrt{n}+1}{\sqrt{n^2+3}}$$

and $\sum \frac{1}{2\sqrt{n}}$ diverges

$\therefore \sum \frac{\sqrt{n}+1}{\sqrt{n^2+3}}$ diverges by D.C.T

$\boxed{15}$ $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

$\rightarrow a_n = \frac{1}{\ln n} \quad , \quad b_n = \frac{1}{n}$

we know $\ln n \leqslant n$

$\rightarrow \frac{1}{\ln n} \geqslant \frac{1}{n}$

and $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges «p-series with $p=1$»

$\therefore \sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges by D. C.T.

يجوز حلها على L.C.T

$\boxed{18}$ $\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$

$\longrightarrow a_n = \frac{3}{n+\sqrt{n}} \quad , \quad b_n = \frac{3}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3/(n+\sqrt{n})}{3/n} = \lim_{n\to\infty} \frac{n}{n+\sqrt{n}} = \lim_{n\to\infty} \frac{\cancel{n}}{n(1+\frac{1}{\sqrt{n}})}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{\sqrt{n}}} = 1 > 0$$

$\therefore \sum a_n$ and $\sum b_n$ both converge or both diverge.

but $\sum b_n = \sum \frac{3}{n}$ diverges «p-series with $p=1$»

$\longrightarrow \sum \frac{3}{n+\sqrt{n}}$ diverges by L. C.T

لو جربنا على D. C.T لانستطيع الحكم لأن

$\frac{3}{n+\sqrt{n}} \geqslant \frac{3}{n}$

$\boxed{27}$ $\displaystyle\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$

$\longrightarrow a_n = \dfrac{1}{\ln(\ln n)}$

$$\ln(\ln n) \leqslant \ln n \leqslant n \qquad \forall n \geq 3$$

$$\frac{1}{\ln(\ln n)} \geqslant \frac{1}{\ln n} \geqslant \frac{1}{n}$$

$$\therefore \frac{1}{\ln(\ln n)} \geqslant \frac{1}{n}$$

and $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n}$ diverges «p-series with $p = 1$»

so $\displaystyle\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$ diverges by D.C.T.

$\boxed{28}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$

ملاحظة $(\ln n)^2 < n$

$\longrightarrow a_n = \dfrac{(\ln n)^2}{n^3}$, $b_n = \dfrac{n}{n^3} = \dfrac{1}{n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(\ln n)^2/n^3}{1/n^2} = \lim_{n\to\infty} \frac{(\ln n)^2}{n}$$

لوبيتال $\frac{\infty}{\infty}$

$$= \lim_{n\to\infty} \frac{2(\ln n)\left(\frac{1}{n}\right)}{1} = \lim_{n\to\infty} \frac{2\ln n}{n} = 0$$

and $\sum b_n = \sum \frac{1}{n^2}$ converges «p-series with $p = 2$»

so $\sum \frac{(\ln n)^2}{n^3}$ converges by L.C.T.

32 $\sum_{n=2}^{\infty} \frac{\ln(n+1)}{n+1}$

Integral test نستطيع حلها ب

$\rightarrow a_n = \frac{\ln(n+1)}{n+1}$, $b_n = \frac{1}{n+1}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln(n+1)/\cancel{(n+1)}}{1/(n+1)} = \lim_{n\to\infty} \ln(n+1) = \infty$

and $\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n+1}$ diverges «harmonic series $\sum \frac{1}{n}$»

$\therefore \sum_{n=1}^{\infty} \frac{\ln(n+1)}{n+1}$ diverges by L.C.T.

43 $\sum_{n=2}^{\infty} \frac{1}{n!}$

$\rightarrow a_n = \frac{1}{n!} = \frac{1}{n(n-1)(n-2)\cdots(2)(1)}$

we know $n! = n(n-1)\cdots(2)(1) \geq n(n-1)$

$\therefore \frac{1}{n!} \leq \frac{1}{n(n-1)}$

and $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ «telescoping series»

$\sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \sum_{n=2}^{\infty} \left(\frac{A}{n} + \frac{B}{n-1}\right)$; $A=-1$, $B=1$

$\therefore S_n = a_2 + a_3 + \cdots + a_n$

$= \left(\frac{-1}{2} + 1\right) + \left(\frac{-1}{3} + \frac{1}{2}\right) + \cdots + \left(\frac{-1}{n} + \frac{1}{n-1}\right)$

$= 1 + \frac{-1}{n}$

$\rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 + \frac{-1}{n}\right) = 1$

$\therefore \sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ converges by nth term test.

$\rightarrow \sum \frac{1}{n!}$ converges by D.C.T

40 $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{3^n + 4^n}$

$\rightarrow a_n = \frac{2^n + 3^n}{3^n + 4^n}$, $b_n = \frac{3^n}{4^n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty}\left(\frac{2^n+3^n}{3^n+4^n} \div \frac{3^n}{4^n}\right)$

$= \lim_{n\to\infty} \frac{(2^n+3^n)\cdot 4^n}{(3^n+4^n)\cdot 3^n}$

$= \lim_{n\to\infty} \frac{8^n + 12^n}{9^n + 12^n}$

$= \lim_{n\to\infty} \frac{12^n\left(\left(\frac{8}{12}\right)^n + 1\right)}{12^n\left(\left(\frac{9}{12}\right)^n + 1\right)} = \boxed{1}$

since $\lim_{n\to\infty} a^n = 0$ if $\alpha a < 1$

and $\sum_{n=1}^{\infty} \frac{3^n}{4^n} = \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$ geometric series $r = \frac{3}{4} < 1$

$\therefore$ converges.

So $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{3^n + 4^n}$ converges by L.C.T.

52 $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$

$\longrightarrow$ let $a_n = \frac{\sqrt[n]{n}}{n^2}$, $b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt[n]{n}/n^2}{1/n^2} = \lim_{n\to\infty} \sqrt[n]{n} = 1$$

So by The limit comparison test:

since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges ((p-series; p=2))

$\therefore \sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$ converges.