- Sec 10.6 : Alternating series, Absolute and Conditional convergence:

* A series in which the terms are alternately positive and negative is an alternating series.
* The alternating series test (Leibniz's test)

$$\sum_{n=1}^{\infty} (-1)^{n+1} U_n$$ converges if:-

1. U_n positive.
2. U_n are eventually nonincreasing $\forall n \geq N$.
3. $U_n \longrightarrow 0$ « $\lim_{n\to\infty} U_n = 0$ ».

* If $\lim_{n\to\infty} U_n \neq 0$ then $\sum (-1)^{n+1} U_n$ diverges by n-th term test.

* If $\sum |a_n|$ converges, then $\sum a_n$ converges absolutely.
* If $\sum a_n$ converges and $\sum |a_n|$ diverges then $\sum a_n$ converges conditionally.

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* If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

* Alternating p-series: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$

converges absolutely if $p > 1$.

Converges conditionally if $0 < p \leq 1$

if $p \leq 0$ diverges.

* Thm: The alternating series estimation:-

If $\sum_{n=1}^{\infty} (-1)^{n+1} U_n$ converges then :-

$S_n = U_1 - U_2 + \dots + (-1)^{n+1} U_n$ approximates L «the sum of the series», with an error «$|E| < U_{n+1}$»

and the sum «L» lies between any $\underline{\underline{2}}$ successive partial sum S_n and S_{n+1}.

إشارة الباقي «remainder» مثل إشارة أول حد غير مستعمل.

*Exercises: page 573

$\boxed{6}$ etermine if the series converges or diverges:

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2+5}{n^2+4}$$ Alternating series:-

$$\lim_{n\to\infty} \frac{n^2+5}{n^2+4} = 1 \neq 0$$

$\therefore \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2+5}{n^2+4}$ diverges by nth term test.

$\boxed{8}$ $\sum_{n=1}^{\infty} (-1)^n \frac{10^n}{(n+1)!}$

① $U_n = \frac{10^n}{(n+1)!}$ +ve, decreasing.

② $\lim_{n\to\infty} U_n = \lim \frac{10^n}{(n+1)!} = \lim \frac{10 \cdot 10 \cdots\cdots 10}{(n+1) \cdot n \cdots\cdots (2)(1)} = \underline{\underline{0}}$

$\therefore \sum_{n=1}^{\infty} (-1)^n \frac{10^n}{(n+1)!}$ converges by alternating series test.

<u>OR</u>: we can check $\sum |a_n|$ and if $\sum |a_n|$ converges then $\sum a_n$ converges.

so $\sum_{n=1}^{\infty} \left| (-1)^n \frac{10^n}{(n+1)!} \right| = \sum_{n=1}^{\infty} \frac{10^n}{(n+1)!}$ «by ratio test»

$$\rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim \frac{10^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n} = \lim \frac{\cancel{10^n} \cdot 10 \cdot \cancel{n!}}{(n+1)\cancel{n!} \cdot \cancel{10^n}} = 0 < 1$$

$\therefore \sum_{n=1}^{\infty} \frac{10^n}{(n+1)!}$ converges so $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{10^n}{(n+1)!}$

$\boxed{13}$ $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}+1}{n+1}$

① $\frac{\sqrt{n}+1}{n+1}$ +ve, decreasing.

② $\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{\sqrt{n}+1}{n+1} = 0$

$\therefore$ by alternating series test, $\sum (-1)^{n+1} \frac{\sqrt{n}+1}{n+1}$ converges.

$\boxed{20}$ Which of the series converges absolutely and which conditionally and which diverge :-

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n!}{2^n}$ alternating series.

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{n!}{2^n} = \lim_{n\to\infty} \frac{n(n-1)\ldots(2)(1)}{2 \cdot 2 \ldots (2)(2)} = \infty \neq 0$$

$\therefore \sum (-1)^{n+1} \frac{n!}{2^n}$ diverges by nth term test

25 $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1+n}{n^2}$ alternating series.

① $U_n = \frac{1+n}{n^2}$ +ve , decreasing.

② $\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{1+n}{n^2} = 0$

$\therefore \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1+n}{n^2}$ converges by alternating series test.

$\rightarrow$ check $\sum_{n=1}^{\infty} \frac{1+n}{n^2}$

« تفحص القيمة المطلقة اذا
كانت converges تكون الأصلية con. abs
واذا كانت diverges تكون الأصلية con cond »

$\frac{1+n}{n^2} > \frac{n}{n^2} = \frac{1}{n}$ and $\sum \frac{1}{n}$ diverges «p-series $P=1$»

so $\sum \frac{1+n}{n^2}$ diverges by D.C.T.

since $\sum a_n$ converges and $\sum |a_n|$ diverges.

$\therefore \sum (-1)^{n+1} \frac{1+n}{n^2}$ converges conditionally.

30 $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n-\ln n}$

① $\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{\ln n}{n-\ln n} = \frac{\infty}{\infty-\infty}$ نحتاج لتعديل لإيجاد limit

$= \lim_{n\to\infty} \frac{1}{\frac{n}{\ln n} - \frac{\ln n}{\ln n}} = \lim_{n\to\infty} \frac{1}{\frac{n}{\ln n} - 1} = 0$

② U_n +ve , decreasing.

$\therefore \sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n-\ln n}$ converges by alternating series test.

to check $\sum \frac{\ln n}{n-\ln n}$ نفحص القيمة المطلقة

by D.C.T $\quad n-\ln n < n$

$\to \frac{1}{n-\ln n} > \frac{1}{n}$

$\to \frac{\ln n}{n-\ln n} > \frac{1}{n}$

and $\sum \frac{1}{n}$ diverges, $\therefore \sum \frac{\ln n}{n-\ln n}$ diverges

39 $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{2^n\, n!\, n}$.

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{(2n)!}{2^n\, n!\, n} = \lim_{n\to\infty} \frac{(n+n)!}{2^n\, n!\, n}$$

$$= \lim_{n\to\infty} \frac{(n+n)(n+n-1)\cdots(n+1)\,\cancel{n!}}{2^n\,\cancel{n!}\, n}$$

$$= \lim_{n\to\infty} \frac{\cancel{2}\cancel{n}\,(n+\underline{n-1})\cdots\cdots(n+1)}{\cancel{2}\cdot 2^{n-1}\cdot \cancel{n}}$$

$$= \lim_{n\to\infty} \frac{(n+\underline{n-1})\cdots\cdots(n+1)}{2\quad\cdots\cdots\quad 2}$$

$$= \infty \quad , \neq 0$$

$\therefore \displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{2^n\, n!\, n}$ diverges by nth term test.

42 $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n^2+n} - n\right)$.

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \left(\sqrt{n^2+n} - n\right)$$

$\infty - \infty$

بنحلها بالضرب بالمرافق

$$= \lim_{n\to\infty} \left(\sqrt{n^2+n} - n\right) \cdot \frac{\sqrt{n^2+n} + n}{\sqrt{n^2+n} + n} = \lim \frac{n^2 + n - n^2}{\sqrt{n^2+n} + n}$$

$$= \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n} + n}$$

$$= \lim \frac{n}{\sqrt{n^2(1+\frac{1}{n})}+n}$$

$$= \lim \frac{n}{n\sqrt{1+\frac{1}{n}}+n}$$

$$= \lim \frac{\cancel{n}}{\cancel{n}(\sqrt{1+\frac{1}{n}}+1)} = \frac{1}{2} \neq 0$$

$\therefore \sum (-1)^n (\sqrt{n^2+n} - n)$ diverges by nth term test

$\boxed{50}$ Estimate the magnitude of the error using the first $\underline{4}$ terms to approximate the sum.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{10^n}$$

$$\rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{10^n} = \underline{\frac{1}{10} - \frac{1}{100} + \frac{1}{1000} - \frac{1}{10000}} + \frac{1}{100000} - \cdots$$

$$\therefore |E| < \frac{1}{100000}$$

$\boxed{54}$ determine how many terms should be used to estimate the sum of the entire series with an error of less than 0.001

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1} = \frac{1}{2} - \frac{2}{5} + \frac{3}{10} - \frac{4}{17} + \cdots$$

$$|U_n| < 0.001$$

$$\frac{n}{n^2+1} < 0.001$$

$$\rightarrow \frac{n}{n^2+1} < \frac{1}{1000} \rightarrow 1000n < n^2+1$$

$$n^2 - 1000n + 1 > 0$$

$$\rightarrow n = \frac{1000 \mp \sqrt{1000^2 - 4(1)(1)}}{2(1)}$$

$$= 1 \times 10^{-3} , 999.9$$

$$\therefore n = 1000$$