10.6: Alternating Series, Absolute and Conditional Convergence

A series in which the terms are alternately positive and negative is an alternating series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots+\frac{(-1)^{n+1}}{n}+\cdots$$

$$-2+1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots+\frac{(-1)^{n}}{2^{n}}\cdot 4+\cdots$$

$$1-2+3-4+5-6+\cdots+(-1)^{n+1}\cdot n+\cdots$$

We see from these examples that the nth term of an alternating series is of the form

$$a_n=(-1)^{n+1}u_n \quad \text{or} \quad a_n=(-1)^{n}u_n$$

where $u_n=|a_n|$ is a positive number

We prove the convergence of alternating series. ②

[This test is only for convergence of alternating series and can't be used for divergence]

The Alternating series Test (Leibniz's Test)

The series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots$

converges if all three of the following conditions are satisfied

① The u_n's are all positive

② The positive u_n's are eventually nonincreasing
$u_n \geq u_{n+1}$ $\forall$ $n \geq N$ for some integer N

③ $u_n \to 0$

Example: The alternating harmonic series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

satisfies the 3 conditions above so the series $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n}$ converges

Check!

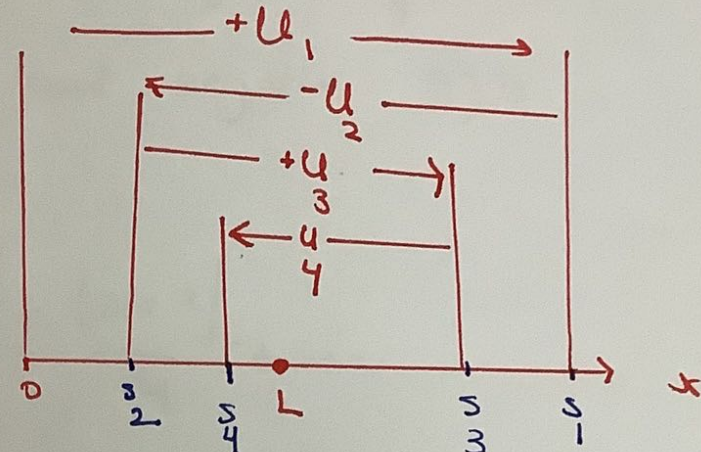
If an alternating series satisfies the three conditions of the Alternating Series test.

$$u_1 - u_2 + u_3 - u_4 + \dots$$

$s_1 = u_1$

$s_2 = u_1 - u_2$

$s_3 = u_1 - u_2 + u_3$



Since $u_2 \leq u_1$ so we dont reach 0 again in step 2 and since the nth term approaches 0 so each backward or forward step is smaller and smaller.

We can conclude that L lies between any two successive sums s_n and s_{n+1} and so differs from s_n by an amount less than u_{n+1}

$$|L - s_n| < u_{n+1}, \text{ for } n \geq N$$

The alternating series Estimation Th.

If the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ satisfies the three conditions of the test then, for $n \geq N$

$$s_n = u_1 - u_2 + u_3 - u_4 + \dots + (-1)^{n+1} u_n$$

approximates the sum L of the series with an error whose absolute value ~~of the first~~. is less than u_{n+1}, the absolute value of the first unused term, Furthermore, the sum L lies between any two successive partial sums s_n and s_{n+1} and the remainder $L - s_n$ has the same as the first unused term.

Example:

$$\sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} + \frac{1}{256} - - -$$

① $|\text{Error}| \leq \frac{1}{256}$ [If we stop after 8 terms]

② the error is the

③ $s_8 = 0.6640625$, $s_9 = 0.66796875$

sum of the geometric series $= \frac{1}{1-\frac{-1}{2}} = \frac{2}{3}$

$0.66796875 > \frac{2}{3} > 0.6640625$

The remainder is $\frac{2}{3} - 0.6640625 = 0.0026041666$

is +ve and less than $\frac{1}{256} = 0.00390625$

Absolute and Conditional Convergence

Definition:

The series $\boxed{\sum a_n}$ $\boxed{\text{converges absolutely}}$ if the corresponding series of absolute values $\boxed{\sum |a_n|}$ converges

Definition:

A series that converges but doesn't converge absolutely $\boxed{\text{converges conditionally}}$

Example.

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{(-1)^{n+1}}{n}$

converges conditionally since it satisfies the 3 condition of AST and it does not converge absolutly

since $\sum \left| \frac{(-1)^{n+1}}{n} \right| = \sum \frac{1}{n}$ divergent harmonic series

Example.

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$$

$\sum \left| (-1)^n \cdot \frac{1}{2^n} \right| = \sum \frac{1}{2^n}$ convergent geometric series

So $\sum (-1)^n \frac{1}{2^n}$ converges absolutely

Absolute Convergence is important for two reasons

1. We have good tests for convergence of series of positive terms
2. If the series converges absolutely, then it converges

Theorem: The absolute Convergence Test

If $\sum |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

Example: $\sum^{\infty} (-1)^{n+1} \cdot \frac{1}{n^2}$

Test for absolute convergence

$$\sum_{1}^{\infty} \left| (-1)^{n+1} \cdot \frac{1}{n^2} \right| \le \sum_{1}^{\infty} \frac{1}{n^2}$$

$\sum \frac{1}{n^2}$ P-series $\rightarrow$ converges

So $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n^2}$ converges absolutely

Example : $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1^2} + \frac{\sin 2}{2^2} + \frac{\sin 3}{3^2} + \frac{\sin 4}{4^2} + \cdots$$

Contains both +ve and negative signs

$$\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right| = \frac{|\sin 1|}{1} + \frac{|\sin 2|}{4} + \cdots$$

$$\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ (convergent P-series)}$$

So $\sum \left|\frac{\sin n}{n^2}\right|$ converges by D.C.T with $\sum_{n=1} \frac{1}{n^2}$

$\rightarrow \sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges abs.

Example: If P is a positive constant, the sequence $\left\{\frac{1}{n^P}\right\}$ is a decreasing seq. with limit zero so

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^P} = 1 - \frac{1}{2^P} + \frac{1}{3^P} - \frac{1}{4^P} + \cdots, \; P > 0$$

converges

If $P > 1$ converges abs.

If $0 < P \le 1$, the series converges conditionally

Rearranging series.

we can rearrange the terms of finite sum and the same result is true for an infinite series that is abs. convergent

Theorem: The rearrangement Th. for abs. converged series

If the series $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $b_1, b_2, \dots, b_n$ is any arrangement of the seq. $\{a_n\}$, then $\sum b_n$ converges absolutely and

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n$$

Example: $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ alternating harmonic series

. it converges ~~absolutely~~ conditionally

$$2L = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 2\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\right)$$

$$= 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + \frac{2}{9} - \cdots$$

$$= (2 - 1) - \frac{1}{2} + \left(\frac{2}{3} - \frac{1}{3}\right) - \frac{1}{4} + \left(\frac{2}{5} - \frac{1}{5}\right) - \frac{1}{6} + \left(\frac{2}{7} - \frac{1}{7}\right) - \frac{1}{8} \cdots$$

(14) $\sum (-1)^{n+1} \dfrac{3\sqrt{n+1}}{\sqrt{n}+1}$

Does this series satisfy the AST?

No

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{3\sqrt{n+1}}{\sqrt{n}+1} = 3 \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n+1}}}{\frac{1}{2\sqrt{n}}}$$

$$= 3 \neq 0$$

There is no need to check the other conditions

(19) $\sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{n}{n^3+1}$

which conv. absolutly, which converge, which diverge?

$$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{n}{n^3+1} \right| = \sum_{n=1}^{\infty} \frac{n}{n^3+1}$$

$\sum \frac{1}{n^2}$ convergent P- series

$$\lim_{n\to\infty} \frac{\frac{n}{n^3+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3}{n^3+1} = 1$$ both converge

So $\sum (-1)^{n+1} \dfrac{n}{n^3+1}$ converges absolutly

29 $\sum_{n=1}^{\infty} (-1)^n \frac{\tan^{-1} n}{1+n^2}$

$\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^2+1}$

Use Integral Test. $f(x) = \frac{\tan^{-1} x}{1+x^2}$ +ve cont. decreasing

$\int \frac{\tan^{-1} x}{1+x^2}\, dx$

$u = \tan^{-1} x$

$du = \frac{dx}{x^2+1}$

~~$I = \int \frac{\tan^{-1} x}{u}\, du$~~

$I = \int u \, du = \frac{u^2}{2}$

Now, $\int_1^{\infty} \frac{\tan^{-1} x}{x^2+1}\, dx = \lim_{a \to \infty} \left. \frac{\tan^{-1} x^2}{2} \right|_1^a$

$= \lim_{a \to \infty} \frac{(\tan^{-1} a)^2 - (\tan^{-1} 1)^2}{2}$

$= \frac{\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2}{2} = \frac{3\pi^2}{32}$

$\sum \frac{\tan^{-1} n}{n^2+1}$ converges by I.T

$\sum (-1)^n \frac{\tan^{-1} n}{n^2+1}$ converges absolutely.