Definition:

A power series about $\boxed{x=0}$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

A power series about $\boxed{x=a}$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots + c_n (x-a)^n + \cdots$$

in which the center a and the coefficients $c_0, c_1, c_2, c_3, \ldots$ are constants

Example:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

This is geometric series with first term $=1$, ratio x

It converges to $\frac{1}{1-x}$ for $|x|<1$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots \quad , \quad -1 < x < 1$$

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad , \quad |x|<1}$$

Example:

$$1-\frac{1}{2}(x-2)+\frac{1}{4}(x-2)^2+\frac{-1}{8}(x-2)^3+\cdots+\left(\frac{-1}{2}\right)^n(x-2)^n+\cdots$$

$a=2$

$c_0=1,\ c_1=-\frac{1}{2},\ c_2=\frac{1}{4},\ \ldots\ldots,\ c_n=\left(\frac{-1}{2}\right)^n$

Geometric: $a=1,\ r=-\frac{1}{2}(x-2)$

$$\text{sum}=\frac{a}{1-r}=\frac{1}{1--\frac{1}{2}(x-2)}=\frac{1}{1+\frac{1}{2}(x-2)}=\frac{1}{1+\frac{x}{2}-1}$$

$$=\frac{1}{\frac{x}{2}}=\frac{2}{x}$$

So

$$\frac{2}{x}=1-\frac{(x-2)}{2}+\frac{(x-2)^2}{4}-\frac{(x-2)^3}{8}+\cdots+\left(\frac{-1}{2}\right)^n(x-2)^n+\cdots$$

$,\ 0<x<4$

Example: For what values of x do the following power series converge? 3.

ⓐ $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} \right| = \lim_{n\to\infty} \left| x \cdot \frac{n}{n+1} \right|$$

$$= \lim_{n\to\infty} |x| \frac{n}{n+1} = |x| \cdot \lim_{n\to\infty} \frac{n}{n+1} = |x| \cdot 1 = |x|$$

* The series converges absolutely for $|x| < 1$

* The series diverges if $|x| > 1$ because the nth term doesn't converge to zero

at $x = 1$

$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n}$ alternating harmonic series which converges conditionally

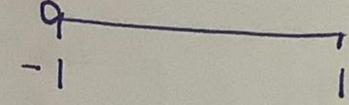
at $x = -1$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot (-1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n} = \sum_{n=1}^{\infty} \frac{(-1)}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$$ Divergent harmonic series

The series converges absolutly for $-1 < X < 1$

The series converges for $-1 < X \leq 1$

The series diverges for $X \leq -1$ or $X > 1$



(b) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{x^{2n+2-1}}{2n+2-1}}{\frac{x^{2n-1}}{2n-1}} \right| = \lim_{n\to\infty} \left| \frac{x^{2n+1}}{2n+1} \cdot \frac{2n-1}{x^{2n-1}} \right|$$

$$= \lim_{n\to\infty} \left| x^2 \frac{2n-1}{2n+1} \right| = \lim_{n\to\infty} |x^2| \left| \frac{2n-1}{2n+1} \right| = |x|^2 \lim_{n\to\infty} \left| \frac{2n-1}{2n+1} \right|$$

$$= |x|^2 \cdot 1 = |x|^2$$

This series converges absolutly for $-1 < X < 1$

at $x = 1$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{2n-1} = \sum_{n=1}^{\infty} (-1)^{n} \cdot \frac{1}{2n-1}$$

converges conditionaly

at $x = -1$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-1)^{2n-1}}{2n-1} = \sum \cancel{(-1)^{3n-2}} = \sum_{n=1}^{\infty} \frac{-(-1)^{n-1}}{2n-1} = \sum_{n=1}^{\infty} \frac{(-1)^{n}}{2n-1}$$

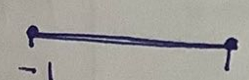
which converges conditionally.

converges absolutly for $-1 < x < 1$

converges conditionally at $x = 1$, $x = -1$

converges on $-1 \leq x \leq 1$

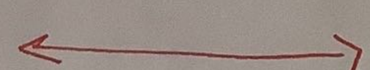
Diverges everywhere.



Example: $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{x \, (n!)}{(n+1)!} \right| = \lim_{n\to\infty} |x| \frac{n!}{(n+1)\, n!} = \lim_{n\to\infty} \frac{|x|}{n+1} \to 0$$

So this series converges for all x

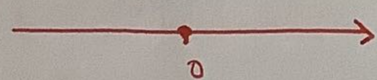


d) $\sum_{n=0}^{\infty} n!\, x^n$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\, x^{n+1}}{n!\, x^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\, n!\, x^{n+1}}{n!\, x^n}\right|$$

$$= \lim_{n\to\infty} (n+1)\,|x| = \infty \quad \text{unless } x=0$$

The series diverges for all values of x except at $X=0$



How To test a power series for convergence

① Use Ratio or Root test to find the interval where the series converges absolutely, ordinarily this is an open interval

$|X-a|<R$ or $a-R < X < a+R$

② If the interval of absolute convergence is finite. test for convergence or divergence at each end point use a comparison test, the integral test or AST

③ If the interval of absolute convergence is $a-R<X<a+R$ the series diverges for $|X-a|>R$ (It doesn't even converges conditionally) because the nth term doesn't approach zero for those values of x.

Operations on Power series

The series Multiplication Theorem for Power Series.

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$. $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converges absoluty for

$|x| < R$ and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \dots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}$$

then $\sum_{n=0}^{\infty} c_n x^n$ Converges abs. ~~for~~ to $A(x) . B(x)$

for $|x| < R$.

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n$$

Example:

$$\left(\sum_{n=0}^{\infty} x^n\right) \cdot \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}\right)$$

$$\left(1 + x + x^2 + x^3 + \dots\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right)$$

$$= a_0 b_0 x^0 + (a_1 b_0 + a_0 b_1) x^1 + (a_2 b_0 + a_1 b_1 + a_0 b_2) x^2 + (a_3 b_0 + a_2 b_1 + a_1 b_2 + a_0 b_3) x^3 + \dots$$

$$= 0 x^0 + 1 . x^1 + \frac{1}{2} x^2 + \frac{5x^3}{6} + - \frac{x^4}{6} \dots$$

$$= x + \frac{x^2}{2} + \frac{5x^3}{6} - \frac{x^4}{6} + \dots$$

Theorem 18: The Convergence Theorem for power series

If the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ converges at $x = c \neq 0$, then it converges absolutly for all x with $|x| < |c|$.

If the series diverges at $x = d$, then it diverges for all x with $|x| > |d|$

Corollary To Th. 18.

The convergence of the series $\sum c_n (x-a)^n$ is described by one of the following three cases

1. There is a positive number R such that the series diverges for x with $\boxed{|x-a| > R}$ but converges absolutly for x with $\boxed{|x-a| < R}$. The series may or may not converges at either of the endpoints. $x = a-R$ and at $x = a+R$

2. The series conveges absolutly for every x $\boxed{R = \infty}$

3. The series converges at $x = a$ and diverges elsewhen $\boxed{R = 0}$

Operations on Power series

Theorem 19: The series Multiplication Theorem for power series

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converges absolutly for $|x|<R$ and

$$\boxed{c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}}$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutly to $A(x)\,B(x)$ for $|x|<R$

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n$$

Theorem:
If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutly for $|x|<R$, then $\sum_{n=0}^{\infty} a_n (f(x))^n$ converges absolutly for any continuous function f on $|f(x)|<R$

Example:

$$\left(\sum_{n=0}^{\infty} x^n\right)\cdot\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}\right)$$

$$= \left(1 + x + x^2 + x^3 + x^4 + \cdots\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)$$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right) + \left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} \cdots\right) + \left(x^3 - \frac{x^4}{2} + \frac{x^5}{3} - \cdots\right) + \cdots$$

$$= x + \frac{x^2}{2} + \frac{5x^3}{6} - \frac{x^4}{6} + \cdots$$

C

Theorem 21: Term by Term Differentiation Theorem

If $\sum c_n(x-a)^n$ has radius of convergence $R>0$, it defines a function

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{on the interval} \quad a-R < x < a+R$$

This function f has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series term by term

$$f'(x) = \sum_{n=0}^{\infty} n c_n (x-a)^{n-1}$$

$$f''(x) = \sum_{n=0}^{\infty} n(n-1) c_n (x-a)^{n-2}$$

$\vdots$

and so on, each of these derived series converges at every point of the interval $a-R < x < a+R$.

d

Example: Find the series for $f'(x)$ and $f''(x)$ if

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots$$

$$f(x) = \sum_{n=0}^{\infty} x^n , \quad -1 < x < 1$$

$$* \; f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots + n x^{n-1} + \dots$$

$$= \sum_{n=0}^{\infty} n x^{n-1} , \quad -1 < x < 1$$

$$* \; f''(x) = \frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + \dots + n(n-1) x^{n-2} + \dots$$

$$= \sum_{n=2}^{\infty} n(n-1) x^{n-2} , \quad -1 < x < 1$$

Term by Term Integration theorem:

suppose that $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ converges

for $a-R < x < a+R$ $(R>0)$. Then

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

converges for $a-R < x < a+R$ and

$$\boxed{\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}} \quad \text{for} \quad a-R < x < a+R$$

e

Example

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \quad -1 \le x \le 1$$

$$f'(x) = 1 - x^2 + x^4 - x^6 + \cdots$$

$$= \frac{1}{1 - - x^2} = \frac{1}{1+x^2}$$

$$f(x) = \int f'(x)\,dx = \int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$$

$f(0) = 0$

So $f(0) = 0 = \tan^{-1} 0 + C \rightarrow C = 0$

$$f(x) = \tan^{-1} x$$

So $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \tan^{-1} x, \quad -1 < x < 1$

Example:

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots, \quad -1 < t < 1$$

Integrate both sides

$$\ln|1+x| = \left. t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \right]_0^x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

or $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad -1 < x < 1$