- Sec 10.8 : Taylor and Maclaurin Series.

- **Definition:**

Let f be a function with derivatives of all orders, then the Taylor series generated by f at $x=a$ is:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)\,(x-a)^k}{k!} = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)\,(x-a)^n}{n!} + \cdots$$

* The Taylor series generated by f at $x=0$ is called the Maclaurin series.

- **Def:** The Taylor polynomial of order n generated by f at $x=a$ is the polynomial:

$$P_n(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)\,(x-a)^n}{n!}.$$

*Exercises: page 606

3 find the Taylor polynomials of orders 1, 2 and 3 generated by f at a.

$f(x) = \ln x \quad ; \quad a = 1.$

→ The Taylor polynomial of order 1:-

$f(1) = \ln 1 = 0$

$f'(x) = \frac{1}{x} \to f'(1) = \frac{1}{1} = 1$

$$P_1(x) = f(1) + f'(1)(x-1)$$
$$= \ln 1 + 1(x-1)$$
$$= (x-1).$$

→ The Taylor polynomial of order 2 :-

$$P_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!} \;;\; f''(x) = \frac{-1}{x^2}$$
$$= 0 + 1(x-1) + \frac{-1(x-1)^2}{2!}$$
$$= x - 1 - \frac{1}{2}(x^2 - 2x - 1)$$
$$= -\frac{1}{2}x^2 + 3x - \frac{1}{2}$$

→ The Taylor polynomial of order 3 :

$$P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!} + \frac{f'''(1)(x-1)^3}{3!} \;;\; f'''(x) = \frac{2}{x^3}$$
$$= (x-1) + \frac{(x-1)^2}{2!} + \frac{2(x-1)^3}{3!}$$
$$= (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3}$$

$\boxed{14}$ Find the Maclaurin series:-

$$\frac{2+x}{1-x}.$$

→ The Taylor series at $x=0$:-

$$f(0)+f'(0)x+\frac{f''(0)x^2}{2!}+\cdots$$

$$=2+3x+\frac{6}{2!}x^2+\cdots$$

$$=2+3x+3x^2+\cdots$$

$$f'(x)=\frac{(1-x)(1)-(2+x)(-1)}{(1-x)^2}=\frac{3}{(1-x)^2}$$

$$f''=\frac{-6(-1)}{(1-x)^3}=\frac{6}{(1-x)^3}$$

$\boxed{20}$ $\sinh x=\dfrac{e^x-e^{-x}}{2}$

→ The Taylor series about $x=0$:-

$$f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)x^3}{3!}+\cdots$$

$$=0+x+0+\frac{x^3}{3!}+\cdots$$

$$=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$$

$f(x)=\sinh x$
$\to f(0)=\sinh 0=0$

$f'(x)=\cosh x$
$\to f'(0)=\cosh 0=1$

$f''(x)=\sinh x$
$\to f''(0)=0$
$\vdots$

22 $\dfrac{x^2}{x+1}$

$\rightarrow$ The Taylor series about $x=0$:-

$$f(0)+\frac{f'(0)\,x}{1!}+\frac{f''(0)\,x^2}{2!}+\cdots$$

$$=0+0+\frac{x^2}{2!}+\frac{-6x^3}{3!}$$

$$=x^2-x^3+\cdots=\sum_{n=2}^{\infty}x^n$$

$f(0)=0$

$\rightarrow f'(x)=\dfrac{(x+1)(2x)-x^2(1)}{(x+1)^2}$

$=\dfrac{x^2+2x}{(x+1)^2}$

$f'(0)=0$

$\rightarrow f''(0)=2$

$\rightarrow f'''(0)=-6$

27 Find the Taylor series generated by f at $x=a$:-

$f(x)=\dfrac{1}{x^2}$; $a=1$

$$f(1)+f'(1)(x-1)+\frac{f''(1)(x-1)^2}{2!}+\cdots$$

$$=1-2(x-1)+\frac{6}{2!}(x-1)^2+\cdots$$

$\rightarrow f(1)=1$

$f'(x)=\dfrac{-2}{x^3}$

$\rightarrow f'(1)=-2$

$f''(x)=\dfrac{6}{x^4}$

$\rightarrow f''(1)=6$

32 $f(x)=\sqrt{x+1}$; $a=0$

$$f(0)+f'(0)\,x+\frac{f''(0)(x)^2}{2!}+\cdots$$

$$=1+\frac{1}{2}x+\frac{\left(\frac{-1}{4}\right)x^2}{2!}+\cdots$$

$$=1+\frac{1}{2}x-\frac{1}{8}x^2+\cdots$$

- Sec 10.9 : Convergence of Taylor Series.

*** Taylor's Theorem:**

If f and its first n derivatives "$f', f'', \dots, f^{(n)}$" are continuous on $[a, b]$, then there is c between a and b such that:

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

In general:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x) \;;\; R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

, for some c between a and x.

If $R_n(x) \longrightarrow 0$ as $(n \to \infty)$, we say that the Taylor series generated by f at $x=a$ converges to f.

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

* The Remainder Estimation Theorem: If $|f^{(n+1)}(t)| \leqslant M$

$$|R_n(x)| \leqslant \frac{M\,|x-a|^{n+1}}{(n+1)!} \;;$$

* Frequently used Taylor series:

① $$\frac{1}{1-x} = 1+x+x^2+\cdots+x^n+\cdots = \sum_{n=0}^{\infty} x^n \;; \quad -1<x<1$$

② $$\frac{1}{1+x} = 1-x+x^2-\cdots+(-x)^n+\cdots = \sum_{n=0}^{\infty} (-1)^n x^n ; \quad -1<x<1$$

③ $$e^x = 1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$ for all x.

④ $$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+\frac{(-1)^n\,x^{2n+1}}{(2n+1)!}+\cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \;;$$ for all x.

⑤ $$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots+\frac{(-1)^n\,x^{2n}}{(2n)!}+\cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \;;$$ for all x.

⑥ $$\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+\frac{(-1)^{n-1}x^{2n}}{(2n)!}+\cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n} \;;$$ $-1<x\leqslant 1$

7 $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots =$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \; ; \; -1 \leq x \leq 1$$

* Exercises: page 613

$\boxed{10}$ Find the Taylor series at $x=0$ of the function

$$\frac{1}{2-x}.$$

→ We know the Taylor series at $x=0$ of $\frac{1}{1-x}$ is

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

→ $\frac{1}{2-x} = \frac{1}{2(1-\frac{x}{2})} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$. Taylor series of $\frac{1}{2-x}$ at $\boxed{x=0}$

$$= \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} x^n .$$

ملاحظة لو كانت at $x=1$:

$$\frac{1}{2-x} = \frac{1}{1+1-x} = \frac{1}{1-(x-1)} = \sum_{n=0}^{\infty} (x-1)^n$$

لا يجوز الحل بهذه الطريقة at $\boxed{x=0}$

12 Use power series operations to find the Taylor series at $x=0$ for the function:- $x^2 \sin x$.

$$\longrightarrow \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \forall x$$

$$x^2 \sin x = x^2\left(x - \frac{x^3}{3} + \frac{x^5}{5!} - \cdots\right) = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \forall x$$

$$x^2 \sin x = x^3 - \frac{x^5}{3!} + \frac{x^7}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+1)!} \quad \forall x$$

18 $\sin^2 x$.

نعرف أن $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ لكن صعب إيجاد $\sin^2 x$

من هذه الصيغة لذا نحاول كتابة $\sin^2 x$ بصورة

أبسط باستخدام المتطابقات.

$$\sin^2 x = \frac{1-\cos(2x)}{2} = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

$$\longrightarrow \sin^2 x = \frac{1}{2} - \frac{1}{2}\left[1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots\right]$$

$$= \frac{1}{2} - \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2x)^{2n}}{(2n)!}$$

$$\sin^2 x = \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2x)^{2n}}{(2n)!} \quad \forall x$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n (2)^{2n-1} x^{2n}}{(2n)!} \quad \forall x .$$

$\boxed{28}$ $\ln(1+x) - \ln(1-x)$

We know $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$

and $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \longrightarrow -\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$

$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$

$\longrightarrow \ln(1+x) - \ln(1-x) =$

$\left(x - \cancel{\frac{x^2}{2}} + \frac{x^3}{3} - \dots\right) - \left(-x - \cancel{\frac{x^2}{2}} - \frac{x^3}{3} - \dots\right)$

$= 2x + \frac{2x^3}{3} + \frac{2x^5}{5} = \sum_{n=0}^{\infty} \frac{2}{2n+1} x^{2n+1}$

$\boxed{35}$ Estimate the error if $P_3(x) = x - \frac{x^3}{6}$ is used to estimate the value of $\sin x$ at $x = 0.1$.

~~sin~~ $\longrightarrow R_n(x) = \frac{f^{(n+1)}(c)\,(x-a)^{n+1}}{(n+1)!}$ for some c between a and x

$n=3$, $x = 0.1$, $a = 0$, $f(x) = \sin x$.

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{(4)}(x) = \sin x$

$\rightarrow f^{(4)}(c) = \sin c$ for some c between 0 and 0.1

$\therefore R_3(X) = \dfrac{\sin c\,(0.1-0)^4}{4!}$ we Know $\sin c \leq 1 \ \forall c$

$\therefore |R_3(x)| \leq \dfrac{0.1^4}{4!}$ $\therefore$ Error $\leq 4.2 \times 10^{-6}$

$\boxed{37}$ For approximately what values of X can you replace $\sin X$ by $X - \frac{X^3}{6}$ with an error of magnitude no greater than 5×10^{-4}?

$$\sin X = X - \frac{X^3}{3!} + \frac{X^5}{5!} ---$$

by alternating series

$$|E| < \frac{|X|^5}{5!} < 5 \times 10^{-4}$$

$$\therefore |X|^5 < 5(120) \times 10^{-4}$$

$$\rightarrow |X|^5 < 0.06$$

$$\therefore |X| < \sqrt[5]{0.06} = 0.5697..$$

41 The approximation $e^x = 1 + x + \frac{x^2}{2}$ is used when x is small. Use the remainder Estimation to estimate the error when $|x| < 0.1$.

$n = 2$, $x < 0.1$, $a = 0$, $f(x) = e^x$

$f'''(x) = e^x$

$$\rightarrow R_2(x) = \frac{f'''(c)(x-0)^3}{3!} \quad ; \quad a < c < x < 0.1$$

$0 < c < 0.1$

$e^c < M \rightarrow e^c \leq e^{0.1}$

$$\therefore |R_2(x)| = \left|\frac{e^c \cdot x^3}{3!}\right| < \frac{e^{0.1}(0.1^3)}{6} = 1.84 \times 10^{-4}$$

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$e < e^{0.1} < 3^{0.1}$

- Sec 10.10 : The Binomial Series and Applications of Taylor series.

* The Binomial series :

For $-1<x<1$;

$$(1+x)^m = 1+\sum_{k=1}^{\infty}\binom{m}{k}x^k \quad \text{where}$$

$$\binom{m}{1} = m \;, \quad \binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!} \;;\; k \geq 2$$

* Exercises : page. 620

$\boxed{10}$ Find the first 4 terms of the binomial series for the function $\frac{x}{\sqrt[3]{1+x}}$

$$\rightarrow \frac{x}{\sqrt[3]{1+x}} = x(1+x)^{-\frac{1}{3}}$$

$$\therefore (1+x)^{\frac{-1}{3}} = 1+\sum_{k=1}^{\infty}\binom{\frac{-1}{3}}{k}x^k$$

$$= 1+\binom{\frac{-1}{3}}{1}x+\binom{\frac{-1}{3}}{2}x^2+\binom{\frac{-1}{3}}{3}x^3+\cdots$$

$$= 1+\frac{-1}{3}x+\frac{(\frac{-1}{3})(\frac{-1}{3}-1)}{2!}x^2+\frac{(\frac{-1}{3})(\frac{-1}{3}-1)(\frac{-1}{3}-2)}{3!}x^3+\cdots$$

$$= 1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \cdots$$

$$\therefore \quad \frac{x}{\sqrt[3]{1+x}} = x(1+x)^{-\frac{1}{3}} = x\left(1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \cdots\right)$$

$$= x - \frac{1}{3}x^2 + \frac{2}{9}x^3 - \frac{14}{81}x^4 + \cdots$$

16 Use series to estimate the integrals with an error of magnitude less than 10^{-3}. $\int_0^{0.2} \frac{e^{-x} - 1}{x} dx$

$$\rightarrow \int_0^{0.2} \frac{e^{-x} - 1}{x} dx = \int_0^{0.2} \frac{\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right) - 1}{x} dx$$

$$= \int_0^{0.2} \frac{-x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots}{x} dx$$

$$= \int_0^{0.2} -1 + \frac{x}{2!} - \frac{x^2}{3!} + \cdots dx$$

$$= -x + \frac{x^2}{4} - \frac{x^3}{18} + \cdots \Big|_0^{0.2}$$

$$= -0.2 + \underbrace{\frac{(0.2)^2}{4}}_{0.01} - \underbrace{\frac{(0.2)^3}{18}}_{4.4\times 10^{-4}} + \cdots$$

نفحص أي حد عند يصبح أقل من 10^{-3}

we estimate the integral by:-

$$\int_0^{0.2} \frac{e^{-x} - 1}{x} dx \cong -0.2 + \frac{(0.2)^2}{4}$$

$$= -0.19 \qquad \text{with error} < \frac{0.2^3}{18} = 4.4\times 10^{-4}$$

26 find a polynomial that will approximate $F(x)$ throughout the given intervals with an error of magnitude less than 10^{-3}.

$$F(x) = \int_0^x t^2 e^{-t^2}\, dt \quad , \quad [0,1].$$

$$= \int_0^x t^2\left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \frac{t^8}{4!} - \frac{t^{10}}{5!} + \cdots\right) dt$$

$$= \int_0^x t^2 - t^4 + \frac{t^6}{2!} - \frac{t^8}{3!} + \frac{t^{10}}{4!} - \frac{t^{12}}{5!} + \cdots\, dt$$

$$= \frac{t^3}{3} - \frac{t^5}{5} + \frac{t^7}{7.2!} - \frac{t^9}{9.3!} + \frac{t^{11}}{11.4!} - \frac{t^{13}}{13.5!} + \cdots \Big|_0^x$$

$$= \frac{x^3}{3} - \frac{x^5}{5} + \underset{0.07}{\frac{x^7}{7.2!}} - \underset{0.185}{\frac{x^9}{9.3!}} + \underset{3.78\times10^{-3}}{\frac{x^{11}}{11.4!}} - \underset{6.4\times10^{-4}}{\frac{x^{13}}{13.5!}} + \cdots ; \quad 0<x<1$$

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$$\therefore F(x) \cong \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7.2!} - \frac{x^9}{9.3!} + \frac{x^{11}}{11.4!}$$

with an error $< \dfrac{1}{13(5!)} = 6.4\times10^{-4}$

30 Use series to evaluate the limits:-

$$\lim_{x\to0} \frac{e^x - e^{-x}}{x}$$

$$= \lim_{x\to0} \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right)}{x}$$

$$= \lim_{x\to0} \frac{\left(2x + 2\frac{x^3}{3!} + \cdots\right)}{x} = \lim_{x\to0} \frac{\cancel{x}\left(2 + \frac{2}{6}x^2 + \cdots\right)}{\cancel{x}} = \boxed{2}$$

33 $\lim_{y\to 0} \frac{y - \tan^{-1} y}{y^3}$

$$= \lim_{y\to 0} \frac{y - \left(y - \frac{y^3}{3} + \frac{y^5}{5} - \frac{y^7}{7} + \cdots\right)}{y^3}$$

$$= \lim_{y\to 0} \frac{\frac{y^3}{3} - \frac{y^5}{5} + \frac{y^7}{7} - \cdots}{y^3}$$

$$= \lim_{y\to 0} \frac{y^3\left(\frac{1}{3} - \frac{y^2}{5} + \frac{y^4}{7} - \cdots\right)}{y^3} = \boxed{\frac{1}{3}}$$