10.10 The Binomial series and Applications of Taylor series

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① In this section we introduce the binomial series for estimating powers and roots of binomial expression $\boxed{(1+x)^m}$

② we also show how series can be used to evaluate nonelementary integrals and limits that lead to indeterminant forms

③ Taylor series of $\tan^{-1} x$

The Binomial series for powers and Roots.

Taylor series generated by $f(x) = (1+x)^m$ when m is constant

$$1 + mx + \frac{m(m-1)}{2!} \cdot x^2 + \frac{m(m-1)(m-2)\,x^3}{3!} + \cdots$$

$$+ \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!} x^k + \cdots$$

This is called binomial series and it converges absolutuy for $|x| < 1$

$$f(x) = (1+x)^m$$

$$f'(x) = m(1+x)^{m-1}$$

$$f''(x) = m(m-1)(1+x)^{m-2}$$

$$f'''(x) = m(m-1)(m-2)(1+x)^{m-3}$$

$$\vdots$$

$$f^{(k)}(x) = m(m-1)(m-2)\cdots(m-k+1)(1+x)^{m-k}$$

Evaluate these at $x=0$ and substitute into the Taylor series formula.

If m is an integer greater than or equal to zero then the series stops after $(m+1)$ terms

If m is not ~~an~~ a positive integer or zero, the series is infinite and converges for $|x|<1$

The Binomial series

for $-1 < x < 1$

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k$$

where we define

$$\binom{m}{1} = m \quad , \quad \binom{m}{2} = \frac{m(m-1)}{2!}$$

$$\binom{m}{k} = \frac{m(m-1)(m-2)\dots(m-k+1)}{k!} \quad , k \geq 3$$

Example: If $m = -1$

$$\binom{-1}{1} = -1 \quad , \quad \binom{-1}{2} = \frac{-1(-2)}{2!} = 1$$

$$\binom{-1}{k} = \frac{-1(-2)(-3)\cdots(-1-k+1)}{k!} = (-1)^k \left(\frac{k!}{k!}\right) = (-1)^k$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots + (-1)^k x^k + \cdots$$

Example:

$$\sqrt{1+x} \simeq 1 + \frac{x}{2} \quad \text{for } |x| \text{ small}$$

$$(1+x)^{\frac{1}{2}} = 1 + \frac{x}{2} + \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)}{2!}x^2 + \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{3!}x^3$$

$$+ \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)}{4!}x^4 + \cdots$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cancel{x}\,\frac{5x^4}{128} + \cdots$$

$$\sqrt{1-x^2} \simeq 1 - \frac{x^2}{2} - \frac{x^4}{8} - \quad \text{for } |x^2| \text{ small}$$

$$\sqrt{1-\frac{1}{x}} \simeq 1 - \frac{1}{2x} - \frac{1}{8x^2} \quad \text{for } \left|\frac{1}{x}\right| \text{ small that is } |x| \text{ large}$$

Example.

$$x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \frac{x^{16}}{9!} + \cdots = \sin x^2$$

$$(x)^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \cdots$$

Evaluating Nonelementary Integrals

Ex $\int \sin x^2 \, dx$ as a power series

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \frac{x^{18}}{9!} - \cdots$$

$$\int \sin x^2 \, dx = \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \frac{x^{19}}{19 \cdot 9!} - \cdots$$

Example 4: Estimate $\int_0^1 \sin x^2 \, dx$ with an error ≤ 0.001 1

$$\int \sin x^2 \, dx = C + \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \frac{x^{19}}{19 \cdot 9!} - \cdots$$

$$\int_0^1 \sin x^2 \, dx = \frac{1}{3} - \frac{1}{7 \cdot 3!} + \frac{1}{11 \cdot 5!} - \frac{1}{15 \cdot 7!} + \frac{1}{19 \cdot 9!} - \cdots$$

This series alternates so

$E \leq$ ~~a~~ first unused term

$$\frac{1}{7 \cdot 3!} = 0.0238095238$$

$$\frac{1}{11 \cdot 5!} = 0.000757575 < 0.001$$

so $\int_0^1 \sin x^2 \, dx \simeq \frac{1}{3} - \frac{1}{42}$ with error ≤ 0.001

If the error $\leq$ ~~0.00~~ 0.1×10^{-6}

$$\int_0^1 \sin x^2 \, dx \simeq \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} - \frac{1}{75600} + \frac{1}{6894720}$$

with error of about 1.08×10^{-9}

Example:

Find a series for $\tan^{-1} x$ by diff.

$$\frac{d \tan^{-1}(x)}{dx} = \frac{1}{1+x^2} = \frac{a}{1-r}$$

$a = 1$

$r = -x^2$

$$\overline{\left[\tan^{-1}(x)\right]} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots + (-1)^n x^{2n} = \frac{1}{1+x^2}$$

$$\tan^{-1} x = \int \frac{1}{1+x^2}\, dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots$$

so

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + R_n(x)$$

$$R_n(x) = (-1)^n \frac{x^{2n+3}}{(2n+3)}$$

$$|R_n(x)| \le \frac{|x|^{2n+3}}{2n+3}$$

If $|x| < 1 \longrightarrow R_n(x) \longrightarrow 0$

so $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \; |x| \le 1$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots + \frac{(-1)^n}{2n+1} + \cdots$$

This is called Leibniz's formula

Evaluating Indeterminate forms

Example: Evaluate

$$\lim \frac{\ln x}{x-1}$$

$$(\ln 1+x)' = \frac{1}{x+1} = \frac{1}{1+x} = \frac{a}{1-r}$$

$a = 1$, $r = -x$, $|r| = |-x| < 1$

~~$\ln(1+x)$~~.

$$(\ln(1+x))' = \frac{1}{1+x} = 1 - x + x^2 - x^3 - x^4 + \cdots$$

Now we have $\ln x$

So replace x by $x-1$

~~$(\ln(1+x))$~~

$$(\ln x)' = \frac{1}{1+x-1} = \frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 - \cdots$$

$$\ln x = \int \frac{1}{x} dx = x - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} - \cdots + c$$

$\ln 1 = 0 \longrightarrow c = -1$

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

$$\frac{\ln x}{x-1} = 1 - \frac{1}{2}(x-1) + \frac{(x-1)^2}{3} - \frac{(x-1)^3}{4} + \cdots$$

$$\boxed{\lim_{x \to 1} \frac{\ln x}{x-1} = 1}$$

Example: Evaluate

$$\lim_{x \to 0} \frac{\sin x - \tan x}{x^3}$$

$$\lim_{x \to 0} \frac{\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots\right] - \left[x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots\right]}{x^3}$$

$$= \lim_{x \to 0} \frac{x^3\left[\frac{-1}{2} - \frac{x^2}{8} - \cdots\right)}{x^3}$$

$$= \lim_{x \to 0} -\frac{1}{2} - \frac{x^2}{8} + \cdots = -\frac{1}{2}$$

Example: Find $\lim\limits_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$

$$\lim_{x\to 0}\frac{x-\sin x}{x\sin x}=\lim_{x\to 0}\frac{x-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}-\cdots\right)}{x\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right)}$$

$$=\lim_{x\to 0}\frac{\frac{x^2}{3!}-\frac{x^4}{5!}-\frac{x^6}{7!}+\cdots}{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots}$$

$$=\lim_{x\to 0}\frac{x^2\left[\frac{1}{3!}-\frac{x^2}{5!}-\frac{x^4}{7!}+\cdots\right]}{x\left[1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\cdots\right]}$$

$$=0\cdot\frac{\frac{1}{3!}}{1}=0$$

If $|x|$ is small

$$\frac{1}{\sin x}-\frac{1}{x}\simeq x\cdot\frac{1}{3!}=\frac{x}{6}\qquad \text{or}\quad \csc x\simeq\frac{1}{x}+\frac{x}{6}$$

Euler Identity

$i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots$$

$$= 1 + \frac{i\theta}{1} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$= \cos\theta \qquad + i\sin\theta$$

$$\boxed{\text{Def: } e^{i\theta} = \cos\theta + i\sin\theta}$$

Look at table 10.1 page 602

v. important

(2) $(1+x)^{\frac{1}{3}}$

write first 4 terms

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)x^3}{3!} + \cdots$$

$$+ \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}x^k + \cdots$$

$$m = \frac{1}{3}$$

$$(1+x)^m = 1 + \frac{1}{3}x + \frac{\frac{1}{3}\left(\frac{-2}{3}\right)}{2!}x^2 + \frac{\frac{1}{3}\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)}{3!}x^3 + \cdots$$

$$\boxed{(1+x)^m = 1 + \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{81}x^3 + \cdots}$$

$\boxed{25}$ Error $\le 10^{-3}$

$$F(x) = \int_0^x \sin t^2 \, dt$$

same as Ex 4

$$F(x) = \int_0^x \left(t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} - \frac{t^{14}}{7!} + \cdots\right) dt$$

$$= \frac{t^3}{3} - \frac{t^7}{7\cdot 3!} + \frac{t^{11}}{11\cdot 5!} - \frac{t^{15}}{15\cdot 7!} + \cdots \Big|_0^x \simeq \frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!}$$

$\cancel{\text{Error}} < \frac{1}{15\cdot 7!} \simeq \cancel{0.0000}13$

(40) The Estimate $\sqrt{1+x} = 1 + \frac{x}{2}$ is used when x is small, Estimate the error when $|x| < 0.01$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots$$

$$|\text{error}| < \left| \frac{x^2}{8} \right| < \frac{(0.01)^2}{8} = 1.25 \times 10^{-5}$$