-Sec 8.7: Improper Integrals التكاملات المعتلة

Up to now, we have required definite integrals to have 2 properties:

جميع التكاملات المشروحة سابقاً كانت
تحقق شرطين أن الفترة منتهية والاقتران متصل عليها

① the domain of integration $[a, b]$ is finite.

② the range of the integrand is finite on $[a, b]$.

-Definition: Integrals with infinite limits of integration are improper integrals of type I.

التكامل المعتل من النوع الأول: أحد الحدود infinite أو كلاهما

① If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

② If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

③ If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$ where c is any real number.

→ If the limit is finite, we say the improper integral converges, and if the limit fails to exist then the improper integral diverges.

* If limit = finite number,
→ improper integral converges.

* If limit $= \infty, -\infty$, DNE
→ improper integral diverges.

- Remark: «p-integrals»

① $\displaystyle\int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \text{converges to } \frac{1}{p-1} & ; p > 1 \\ \text{diverges} & ; p \leq 1 \end{cases}$

② $\displaystyle\int_0^{1} \frac{dx}{x^p} = \begin{cases} \text{converges to } \frac{1}{1-p} & ; p < 1 \\ \text{diverges} & ; p \geq 1 \end{cases}$

*** Definition:** Integrals of functions that become infinite at a point within the interval of integration are improper integrals of type II.

عندما تكون أحد النقاط لا تنتمي لمجال الاقتران وعندها infinite discontinuity

① If $f(x)$ is continuous on $(a, b]$ and discontinuous at $x = a$, then $\displaystyle\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx$

② If $f(x)$ is continuous on $[a, b)$ and discontinuous at $x = b$, then $\displaystyle\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$.

③ If $f(x)$ is discontinuous at c; $a<c<b$ and continuous on $[a,b] \setminus \{c\}$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

→ If the limit is finite, we say the improper integral converges and if the limit does not exist, then the improper integral diverges.

* Tests for Convergence and Divergence:

① Direct Comparison Test: D.C.T

let f and g be continuous on $[a,\infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then:

→ $\int_a^\infty f(x)\,dx$ converges if $\int_a^\infty g(x)\,dx$ converges.

→ $\int_a^\infty g(x)\,dx$ diverges if $\int_a^\infty f(x)\,dx$ diverges.

② Limit Comparison Test: L.C.T

If the positive functions f and g continuous on $[a,\infty)$ and if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$; $0 < L < \infty$

«L: finite number»

then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ both converges or both diverges.

* Exercises: page 487

$\boxed{21}$ Evaluate $\displaystyle\int_{-\infty}^{0} \underbrace{\theta e^{\theta}}_{\text{cont. on } (-\infty,0]} \, d\theta$.

Improper integral of type I.

θ	$+$	e^{θ}
1	$-$	e^{θ}
0		e^{θ}

$$\rightarrow \int_{-\infty}^{0} \theta e^{\theta} \, d\theta = \lim_{a \to -\infty} \int_{a}^{0} \theta e^{\theta} \, d\theta$$

$$= \lim_{a \to -\infty} \left(\theta e^{\theta} - e^{\theta}\right)\Big|_{a}^{0}$$

$$= \lim_{a \to -\infty} (0 - e^{0}) - (a e^{a} - e^{a})$$

$$= \lim_{a \to -\infty} -1 - \underbrace{a e^{a}}_{\substack{-\infty \cdot 0 \\ \text{L'Hopital's rule}}} + \underbrace{e^{a}}_{0}$$

$$= -1 - \lim_{a \to -\infty} \underbrace{\frac{a}{e^{-a}}}_{= \frac{\infty}{\infty}} + 0$$

$$= -1 - \underbrace{\lim_{a \to -\infty} \frac{1}{-e^{-a}}}_{0} \rightarrow \frac{1}{-e^{-\infty}} = \frac{1}{-e^{\infty}} = 0 \quad = -1$$

$\therefore \displaystyle\int_{-\infty}^{0} \theta e^{\theta} \, d\theta$ converges to -1.

4 $\displaystyle\int_0^4 \frac{dx}{\sqrt{4-x}}$

$\dfrac{1}{\sqrt{4-x}}$ is discontinuous at $x=4$.

$$= \lim_{b\to 4^-} \int_0^b \frac{dx}{\sqrt{4-x}} = \lim_{b\to 4^-} \int_0^b (4-x)^{-\frac{1}{2}}\, dx$$

$$= \lim_{b\to 4^-} -\frac{(4-x)^{\frac{1}{2}}}{\frac{1}{2}} \Big|_0^b$$

$$= \lim_{b\to 4^-} -2\sqrt{4-x} \Big|_0^b$$

$$= \lim_{b\to 4^-} \left(-2\sqrt{4-b} - -2\sqrt{4}\right)$$

$$= 0 + 4 = 4.$$

10 $\displaystyle\int_{-\infty}^2 \frac{2\, dx}{x^2+4}$

$\displaystyle\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

$$= \lim_{a\to -\infty} \int_a^2 \frac{2}{x^2+4}\, dx$$

$$= \lim_{a\to -\infty} 2 \cdot \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) \Big|_a^2$$

$$= \lim_{a\to -\infty} \tan^{-1} 1 - \tan^{-1}\left(\frac{a}{2}\right)$$

$$= \frac{\pi}{4} - \lim_{a\to -\infty} \tan^{-1}\left(\frac{a}{2}\right) = \frac{\pi}{4} - -\frac{\pi}{2} = \frac{3\pi}{4}$$

25 $\int_0^1 x \ln x \, dx$

$x \ln x$ is discontinuous at a

$$= \lim_{a\to 0^+} \int_a^1 x \ln x \, dx$$

$$\int x \ln x \, dx$$

let $u = \ln x$, $dv = x$

$du = \frac{1}{x} dx$, $v = \frac{x^2}{2}$

$$\therefore \int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} \cdot \frac{x^2}{2}$$

$$= \lim_{a\to 0^+} \left[\frac{x^2}{2} \ln x - \frac{1}{4} x^2 \right]_a^1$$

$$= \lim_{a\to 0^+} \left[\left(\frac{1}{2} \ln 1 - \frac{1}{4}\right) - \left(\frac{a^2}{2} \ln a - \frac{1}{4} a^2\right) \right]$$

$$= \lim_{a\to 0^+} \left[-\frac{1}{4} - \frac{a^2}{2} \ln a + \frac{1}{4} a^2 \right]$$

$$= \frac{-1}{4} - \lim_{a\to 0^+} \frac{\ln a}{2/a^2} + \lim_{a\to 0} \frac{1}{4} a^2$$

$\frac{-\infty}{\infty}$ H.lopital rule.

$$= \frac{-1}{4} - \lim_{a\to 0^+} \frac{1/a}{-4/a^3} + 0$$

$$= \frac{-1}{4} - \lim_{a\to 0^+} \frac{-a^2}{4} = \boxed{\frac{-1}{4} - 0} = \frac{-1}{4}.$$

32 $\int_0^2 \frac{dx}{\sqrt{|x-1|}}$

$$|x-1| = \begin{cases} x-1 \ ; & x > 1 \\ 1-x \ ; & x \leq 1 \end{cases}$$

$$= \int_0^1 \frac{dx}{\sqrt{1-x}} + \int_1^2 \frac{dx}{\sqrt{x-1}}$$

$$= \lim_{b\to 1^-} \int_0^b \frac{dx}{\sqrt{1-x}} + \lim_{a\to 1^+} \int_a^2 \frac{dx}{\sqrt{x-1}}$$

$$\int \frac{dx}{\sqrt{1-x}} = \int (1-x)^{-\frac{1}{2}} dx = \frac{-(1-x)^{\frac{1}{2}}}{\frac{1}{2}} = -2(1-x)^{\frac{1}{2}}$$

$$\int \frac{dx}{\sqrt{x-1}} = \int (x-1)^{-\frac{1}{2}} dx = 2(x-1)^{\frac{1}{2}}$$

$$= \lim_{b\to 1^-} -2\sqrt{1-x}\,\Big|_0^b + \lim_{a\to 1^+} 2\sqrt{x-1}\,\Big|_a^2$$

$$= \lim_{b\to 1^-}\left(-2\sqrt{1-b}+2\right) + \lim_{a\to 1^+}\left(2-2\sqrt{a-1}\right)$$

$$= \left(-2(0)+2\right) + \left(2-0\right)$$

$$= (0+2) + (2-0) = 4.$$

$\boxed{37}$ $\displaystyle\int_0^{\pi} \frac{\sin\theta}{\sqrt{\pi-\theta}}\,d\theta$

$$0 \leq \sin\theta \leq 1 \quad ; \quad \text{for all} \quad 0 \leq x \leq \pi$$

$$0 \leq \frac{\sin\theta}{\sqrt{\pi-\theta}} \leq \frac{1}{\sqrt{\pi-\theta}}$$

$$0 \leq \int_0^{\pi} \frac{\sin\theta}{\sqrt{\pi-\theta}}\,d\theta \leq \int_0^{\pi} \frac{1}{\sqrt{\pi-\theta}}\,d\theta$$ converges since

$$\longrightarrow \int_0^{\pi} (\pi-\theta)^{-\frac{1}{2}}\,d\theta = \lim_{b\to\pi^-} \int_0^{b} (\pi-\theta)^{-\frac{1}{2}}\,d\theta$$

$$= \lim_{b\to\pi^-} -2(\pi-\theta)^{\frac{1}{2}}\,\Big|_0^b$$

$$= \lim_{b\to\pi^-} -2\sqrt{\pi-\theta}\,\Big|_0^b$$

$$= \lim_{b\to\pi^-}\left(-2\sqrt{\pi-b}+2\sqrt{\pi}\right) = 0 + 2\sqrt{\pi} = 2\sqrt{\pi}$$

$\therefore \displaystyle\int_0^{\pi} \frac{\sin\theta}{\sqrt{\pi-\theta}}\,d\theta$ converges by Direct comparison test

$\boxed{14}$ $$\int_{-\infty}^{\infty} \frac{x}{\underbrace{(x^2+4)^{3/2}}_{\downarrow}}\,dx$$

cont. on $(-\infty,\infty)$.

$$\longrightarrow \int_{-\infty}^{\infty} \frac{x}{(x^2+4)^{3/2}}\,dx = \underbrace{\int_{-\infty}^{0} \frac{x}{(x^2+4)^{3/2}}\,dx}_{1} + \underbrace{\int_{0}^{\infty} \frac{x}{(x^2+4)^{3/2}}\,dx}_{2}$$

but $\int \frac{x}{(x^2+4)^{3/2}}\,dx$

let $u = x^2+4$

$du = 2x\,dx$

$$\longrightarrow \int \frac{du}{2\,u^{3/2}} = \frac{1}{2}\cdot\frac{u^{-1/2}}{-1/2} = \frac{-1}{\sqrt{u}} = \frac{-1}{\sqrt{x^2+4}}$$

$$1\ \int_{-\infty}^{0} \frac{x}{(x^2+4)^{3/2}}\,dx = \lim_{b\to-\infty} \frac{-1}{\sqrt{x^2+4}}\Bigg|_{b}^{0}$$

$$= \lim_{b\to-\infty}\left(\frac{-1}{2} + \frac{1}{\sqrt{b^2+4}}\right)$$

$$= \frac{-1}{2}$$

$$2\ \int_{0}^{\infty} \frac{x}{(x^2+4)^{3/2}}\,dx = \lim_{b\to\infty} \frac{-1}{\sqrt{x^2+4}}\Bigg|_{0}^{b}$$

$$= \lim_{b\to\infty}\left(\frac{-1}{\sqrt{b^2+4}} - \frac{-1}{2}\right) = \frac{1}{2}$$

$$\therefore \int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+4)^{3/2}} = \frac{-1}{2} + \frac{1}{2} = 0$$

so $\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+4)^{3/2}}$ converges to $\underline{\underline{0}}$.

$\boxed{16}$ $\displaystyle\int_0^2 \underbrace{\frac{s+1}{\sqrt{4-s^2}}}\,ds$

↳ discontinuous at $s=2$.

$$\longrightarrow \int \frac{s+1}{\sqrt{4-s^2}}\,ds = \underbrace{\int \frac{s}{\sqrt{4-s^2}}\,ds}_{\text{substitution}} + \underbrace{\int \frac{1}{\sqrt{4-s^2}}\,ds}_{a=2}$$

substitution

$4-s^2=u$

$-2s\,ds=du$

$$= \int \frac{du}{-2\sqrt{u}} + \sin^{-1}\left(\frac{s}{2}\right)$$

$$= \frac{-1}{2}\int u^{-\frac{1}{2}}\,du + \sin^{-1}\left(\frac{s}{2}\right)$$

$$= \frac{-1}{2}\cdot\frac{u^{\frac{1}{2}}}{1/2} + \sin^{-1}\left(\frac{s}{2}\right)$$

$$= -\sqrt{4-s^2} + \sin^{-1}\left(\frac{s}{2}\right)$$

Now $\displaystyle\int_0^2 \frac{s+1}{\sqrt{4-s^2}}\,ds = \lim_{b\to 2^-}\int_0^b \frac{s+1}{\sqrt{4-s^2}}\,ds$

$$= \lim_{b\to 2^-}\left(-\sqrt{4-s^2} + \sin^{-1}\left(\frac{s}{2}\right)\right)\Big|_0^b$$

$$= \lim_{b\to 2^-}\left[-\sqrt{4-b^2} + \sin^{-1}\left(\frac{b}{2}\right)\right] - \left(-\sqrt{4}+0\right)$$

$$= -(0) + \sin^{-1}(1) + 2$$

$$= \frac{\pi}{2} + 2$$

$\therefore \displaystyle\int_0^2 \frac{s+1}{\sqrt{4-s^2}}\,ds$ converges to $\dfrac{\pi+4}{2}$

41 Test the integral for convergence

$$\int_0^{\pi} \frac{dt}{\sqrt{t}+\sin t}$$

$\longrightarrow$ $\sin t \geq 0$; $t \in [0, \pi]$

$\sqrt{t} + \sin t \geq \sqrt{t}$ +ve.

$$\frac{1}{\sqrt{t}+\sin t} \leq \frac{1}{\sqrt{t}} \quad (*)$$

Now we want to check $\int_0^{\pi} \frac{1}{\sqrt{t}} dt$

discontinuous at $t=0$.

$$\int_0^{\pi} t^{-\frac{1}{2}} dt = \lim_{b \to 0^+} \int_b^{\pi} t^{-\frac{1}{2}} dt$$

$$= \lim_{b \to 0^+} 2\sqrt{t} \Big|_b^{\pi}$$

$$= \lim_{b \to 0^+} \left(2\sqrt{\pi} - 2\sqrt{b}\right)$$

$$= 2\sqrt{\pi} - 0 = 2\sqrt{\pi}$$

$\therefore \int_0^{\pi} \frac{1}{\sqrt{t}} dt$ converges and from $(*)$

the integral $\int_0^{\pi} \frac{dt}{\sqrt{t}+\sin t}$ converges by D.C.T

50 $\int_0^\infty \frac{d\theta}{1+e^\theta}$

improper integral of type I.

$e^\theta \leq 1+e^\theta$

$\frac{1}{e^\theta} \geq \frac{1}{1+e^\theta}$ ⊛

Now we want to check $\int_0^\infty \frac{1}{e^\theta} d\theta$:-

$$\int_0^\infty \frac{1}{e^\theta} d\theta = \lim_{b\to\infty} \int_0^b e^{-\theta} d\theta$$

$$= \lim_{b\to\infty} -e^{-\theta} \Big|_0^b$$

$$= \lim_{b\to\infty} \left(-e^{-b} + e^{0}\right)$$

$$= 0 + e^0 = 1$$

$\therefore \int_0^\infty \frac{1}{e^\theta} d\theta$ converges to 1, and from ⊛ :-

$\int_0^\infty \frac{1}{1+e^\theta} d\theta$ converges by D.C.T

58 $\int_2^\infty \frac{1}{\ln x} dx$

improper integral of type I:

$\ln x \leq x \qquad x \in [2, \infty)$

$\frac{1}{\ln x} \geq \frac{1}{x}$ ⊛

$$\int_2^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} \ln x \Big|_2^b$$

$$= \lim_{b\to\infty} (\ln b - \ln 2)$$

$$= \infty$$

$\therefore \int_2^{\infty} \frac{1}{x}\,dx$ diverges and from ⊛ :-

$\int_2^{\infty} \frac{1}{\ln x}\,dx$ diverges by D.C.T

62 $\int_1^{\infty} \frac{1}{e^x - 2^x}\,dx$

let $g(x) = \frac{1}{e^x}$

Now $\lim_{x\to\infty} \frac{1/e^x}{1/(e^x - 2^x)} = \lim_{x\to\infty} \frac{e^x - 2^x}{e^x}$

$$= \lim_{x\to\infty} \left(\frac{e^x}{e^x} - \frac{2^x}{e^x}\right)$$

$$= \lim_{x\to\infty} \left(1 - \left(\frac{2}{e}\right)^x\right)$$

$$= 1 - 0 = 1 \quad ⊛$$

Now $\int_1^{\infty} \frac{1}{e^x}\,dx = \lim_{b\to\infty} \int_1^b e^{-x}\,dx$

$$= \lim_{b\to\infty} -e^{-x} \Big|_1^b$$

$$= \lim_{b\to\infty} (-e^{-b} + e^{-1}) = e^{-1}$$

$\therefore \int_1^\infty \frac{1}{e^x}\,dx$ converges, and from ⊛ :

$\int_1^\infty \frac{dx}{e^x - 2^x}$ converges by L.C.T

65 Find the values of p for which each integral converges:

a) $\int_1^2 \frac{dx}{x(\ln x)^p}$

discontinuous at $x = 1$ since $\ln 1 = 0$

$\underline{p \neq 1} \rightarrow \int_1^2 \frac{dx}{x(\ln x)^p} = \lim_{a \to 1^+} \int_a^2 \frac{dx}{x(\ln x)^p}$

$= \lim_{a \to 1^+} \frac{(\ln x)^{-p+1}}{-p+1} \Big|_a^2$

$= \lim_{a \to 1^+} \left[\frac{(\ln 2)^{-p+1}}{-p+1} - \frac{(\ln a)^{-p+1}}{-p+1} \right]$

$= \begin{cases} \frac{(\ln 2)^{-p+1}}{-p+1} - \lim_{a \to 1^+} \frac{1}{(-p+1)(\ln a)^{p-1}} = -\infty \; ; \; p > 1 \\ \frac{(\ln 2)^{-p+1}}{-p+1} - \lim_{a \to 1^+} \underbrace{\frac{(\ln a)^{-p+1}}{-p+1}}_{\to 0} = \frac{(\ln 2)^{-p+1}}{-p+1} \; ; \; p < 1 \end{cases}$

$p \neq 1$

$\int \frac{dx}{x(\ln x)^p} =$

let $u = \ln x$

$du = \frac{1}{x}dx$

$\int \frac{du}{u^p} = \frac{u^{-p+1}}{-p+1}$

$= \frac{(\ln x)^{-p+1}}{-p+1}$

$\underline{p = 1} \rightarrow \int_1^2 \frac{dx}{x(\ln x)} = \lim_{a \to 1^+} \int_a^2 \frac{dx}{x \ln x}$

$= \lim_{a \to 1^+} \ln(\ln x) \Big|_a^2$

$\int \frac{dx}{x \ln x} =$

let $u = \ln x$

$du = \frac{dx}{x}$

$\int \frac{du}{u} = \ln u$

$= \ln(\ln x)$

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$$= \ln(\ln 2) - \lim_{a \to 1^+} \ln\underbrace{(\ln a)}_{\ln(\ln 1^+) = \ln 0^+ = \infty}$$

$$= -\infty$$

$$\therefore \int_1^2 \frac{1}{x(\ln x)^p}\,dx = \begin{cases} \text{converges to } \dfrac{(\ln 2)^{-p+1}}{-p+1} & ; \; p < 1 \\ \text{diverges} & ; \; p \geqslant 1 \end{cases}$$

b) $\displaystyle\int_2^\infty \frac{dx}{x(\ln x)^p}$

$\longrightarrow$ $p < 1$:

$$\int_2^\infty \frac{dx}{x(\ln x)^p} = \lim_{b\to\infty} \int_2^b \frac{dx}{x(\ln x)^p}$$

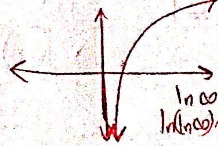
$$= \lim_{b\to\infty} \left.\frac{(\ln x)^{-p+1}}{-p+1}\right|_2^b$$

$$= \lim_{b\to\infty} \frac{(\ln b)^{-p+1}}{-p+1} - \frac{(\ln 2)^{-p+1}}{-p+1}$$

"$-p+1$: +ve" $(\ln b)^{-p+1} \to (\ln \infty)^{-p+1} \to \infty$

$$= \infty$$

$\longrightarrow$ $p = 1$:



$\ln\infty \to \infty$

$\ln(\ln\infty) \to \infty$

$$\int_2^\infty \frac{dx}{x \ln x} = \lim_{b\to\infty} \ln(\ln x)\Big|_2^b$$

$$= \lim_{b\to\infty} \ln(\ln b) - \ln(\ln 2) = \infty$$

$\longrightarrow p > 1:$

$$\int_2^\infty \frac{dx}{x(\ln x)^p} = \lim_{b\to\infty} \frac{(\ln x)^{-p+1}}{-p+1} \Bigg|_2^\infty$$

$$= \lim_{b\to\infty} \frac{(\ln b)^{-p+1}}{-p+1} - \frac{(\ln 2)^{-p+1}}{-p+1}$$

$$= \lim_{b\to\infty} \frac{1}{(-p+1)(\ln b)^{p-1}} - \frac{(\ln 2)^{-p+1}}{-p+1}$$

$$= \quad 0 - \frac{(\ln 2)^{-p+1}}{-p+1}$$

$$= \frac{1}{(p-1)(\ln 2)^{p-1}}$$

$$\therefore \int_2^\infty \frac{dx}{x(\ln x)^p} = \begin{cases} \text{converges to } \dfrac{1}{(p-1)(\ln 2)^{p-1}} & ; \; p > 1 \\ \text{diverges} & ; \; p \leq 1 \end{cases}$$