10.6 Alternating Series, Absolute and Conditional Convergence

Note Title ٢٠٢٣/٠٥/٢٤

Def: A series in which the terms are alternately positive and negative is an **alternating series**.

Here are three examples:

$$1)\ \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots+\frac{(-1)^{n+1}}{n}+\cdots \qquad (1)$$

$$2)\ \sum_{n=1}^{\infty}\frac{(-1)^n 4}{2^n} = -2+1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots+\frac{(-1)^n 4}{2^n}+\cdots \qquad (2)$$

$$3)\ \sum_{n=1}^{\infty}(-1)^{n+1}n = 1-2+3-4+5-6+\cdots+(-1)^{n+1}n+\cdots \qquad (3)$$

We see from these examples that the nth term of an alternating series is of the form

$$a_n = (-1)^{n+1}u_n \qquad \text{or} \qquad a_n = (-1)^n u_n$$

where $u_n = |a_n|$ is a positive number.

المتسلسلة الأولى تسمى Alternating harmonic series وسوف نرى بعد قليل أنها تقاربية
المتسلسلة الثانية هي متسلسلة هندسية فيها $r = -\frac{1}{2}$ لذا فهي تقاربية .
المتسلسلة الثالثة تباعدية لأنه $a_n \not\to 0$.

We prove the convergence of the alternating harmonic series by applying the Alternating Series Test. The Test is for convergence of an alternating series and cannot be used to conclude that such a series diverges. اختبار (AST) يفحص تقارب المتسلسلات المتذبذبة.

THEOREM 14—The Alternating Series Test (Leibniz's Test) The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if all three of the following conditions are satisfied:

1. The u_n's are all positive.
2. The positive u_n's are (eventually) nonincreasing: $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.
3. $u_n \to 0$.

Example: (The Alternating harmonic series)

For the Series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

$u_n = \frac{1}{n}$. Clearly

1) $u_n > 0 \quad \forall n \geq 1$

2) $u_n' = \frac{-1}{n^2} < 0$ so $u_n \searrow$

3) $u_n \longrightarrow 0$,

So by AST, $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n}$ converges.

ملاحظات هامة: ① بشكل عام ، النظرية السابقة تستخدم لفحص التقارب للمتسلسلات المتذبذبة. إذا حققت الشروط ، ولكن إذا اختل شرط فإنه لا يستنتج فوراً أنه متسلسلة متباعدة.

② بالتحليل الدقيق للشروط الثلاثة نجد أنه :

أ- الشرط الأول : $u_n's \geq 0$ نضمن أن تكون المتسلسلة متذبذبة ، وبالتالي إذا لم يتحقق فإننا لا نضمن تذبذب المتسلسلة.

ب- الشرط الثاني : أن تكون u_n nonincreasing $(u_n \geq u_{n+1})$ ، وإذا لم يتحقق الشرط فهذا لا يعني أنها تزايدية ، ولذا إذا اختل هذا الشرط فإن النظرية لا تطبق ، ورغم ذلك لا يصح أن نستنتج أن المتسلسلة متباعدة.

جـ- الشرط الثالث : $u_n \longrightarrow 0$ ، يضمن أن $a_n \longrightarrow 0$ ، وإذا لم يتحقق (الشرط) فإننا نستطيع إثبات أن $a_n \not\longrightarrow 0$ ، وبالتالي ستكون المتسلسلة متباعدة.

③ من الملاحظة السابقة ، إذا اختل الشرط الثاني ، فهذا لا يعني أن u_n تزايدية ولكن إذا أثبتنا أن $u_n \leq u_{n+1}$ لكل n ، فإن هذا يعني أن

$$u_1 \leq u_2 \leq u_3 \leq \cdots$$

وبالتالي فإن $u_n \geq u_1$ لكل n ، عليه فإنه بالتأكيد $u_n \not\longrightarrow 0$
لذا من هذه الحالة يمكن أن نستنتج أن المتسلسلة متباعدة.

Absolute and Conditional Convergence

DEFINITION A series $\sum a_n$ **converges absolutely** (is **absolutely convergent**) if the corresponding series of absolute values, $\sum |a_n|$, converges.

Examples: 1) The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges abs., since the series of abs. values $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

2) The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is not converge abs. since $\sum \frac{1}{n}$ div.

DEFINITION A series that converges but does not converge absolutely **converges conditionally.**

Example: Since the alternating harmonic series converges but does not converge absolutely, so it converges conditionally.

Thrm: (The absolute convergence test)

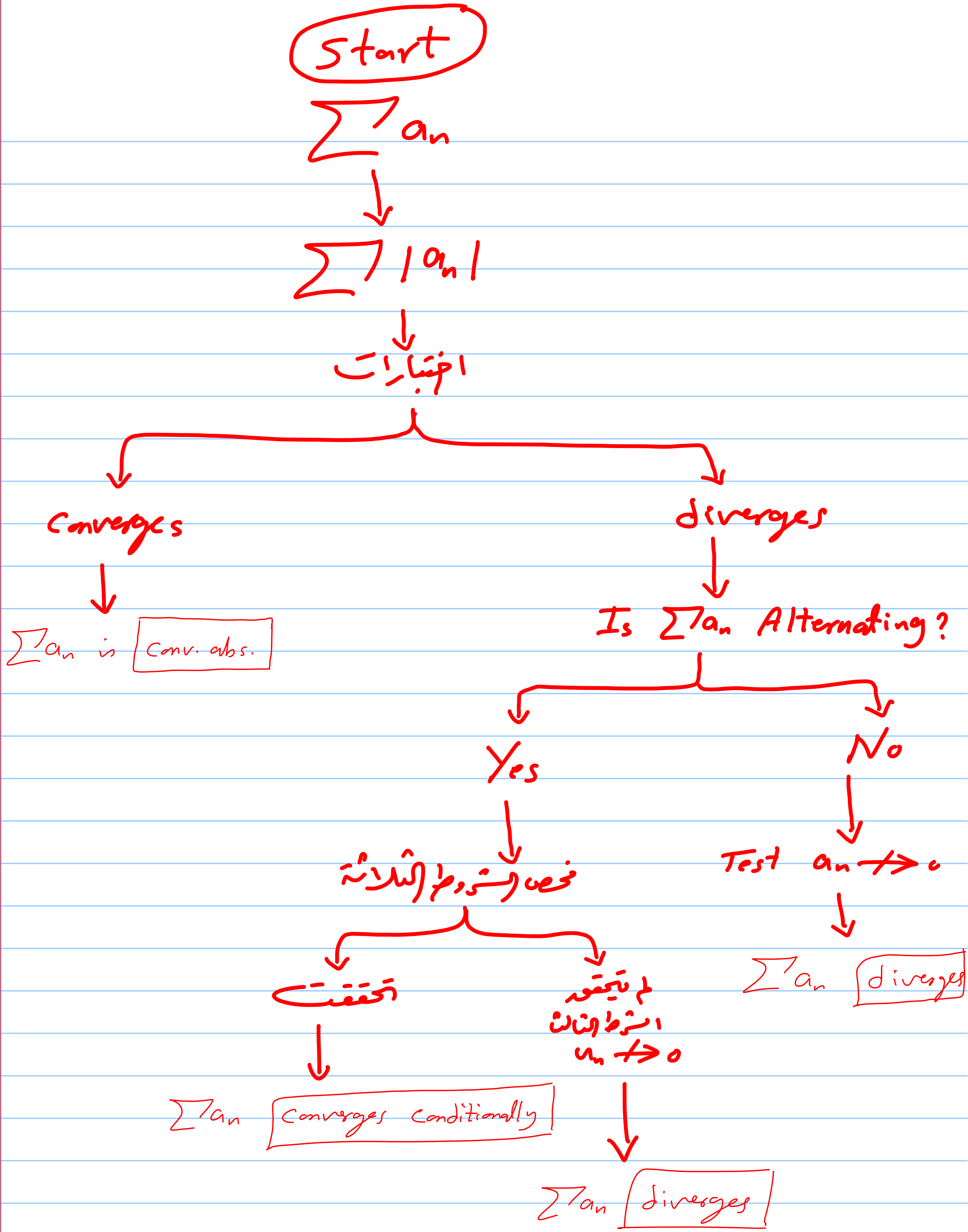
If the series of absolute values $\sum |a_n|$ converges, then the series $\sum a_n$ converges

That is; Converge absolutely $\Longrightarrow$ Converge.

ملحوظة: عكس النظرية غير صحيح / فقد تكون المتسلسلة conv. لكنها ليست conv. abs. / ومثال ذلك $\sum (-1)^n \frac{1}{n}$ تقاربية لكنها ليست تقاربية تقارب مطلق.

Example: The series $\sum \frac{(-1)^{n+1}}{n^2}$ converges, since it converges abs.

المخطط التالي يوضح كيفية التعامل مع المتسلسلات عندما تحتوي على حدود سالبة.



Examples: Determine whether the following series conv., conv. abs., conv. conditionally, or diverge:

1) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ (Note that the series is not alternating)

Sol: Consider the series of abs. values $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$

$0 \leq |\sin n| \leq 1 \quad \forall n \implies 0 \leq \frac{|\sin n|}{n^2} \leq \frac{1}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges so by DCT, $\sum \frac{|\sin n|}{n^2}$ converges and hence $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges absolutely.

2) (The alternating p-series)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots$$

Sol: The series of abs. values $\sum \frac{1}{n^p}$ which converges if $p > 1$ and diverges if $0 < p \leq 1$. So the alternating p-series converges absolutely if $p > 1$.

If $0 < p \leq 1$, the series $\sum \frac{(-1)^{n+1}}{n^p}$ is not conv. abs.,

but $u_n = \frac{1}{n^p}$

1) $u_n \geq 0 \quad \forall n$

2) $u_n' = \frac{-p}{n^{1+p}} < 0 \implies u_n \searrow$

3) $u_n \longrightarrow 0$

So by AST, the alternating p-series conv. cond.

الخلاصة

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$$

→ converges abs. if $p > 1$

→ conv. cond. if $0 < p \leq 1$.

3) $\sum_{n=1}^{\infty} (-1)^n \ln\left(1 + \frac{1}{n}\right)$

Sol: The series of abs. values is $\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right)$

$[\ \ln\left(1 + \frac{1}{n}\right) > 0 \quad \forall n \geq 1\ ]$

Take $a_n = \ln\left(1+\frac{1}{n}\right)$, $\quad b_n = \frac{1}{n}$, $\quad \sum b_n$ diverges.

Consider $\displaystyle \lim \frac{a_n}{b_n} = \lim \frac{\ln\left(1+\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} \left(\frac{0}{0}\right) \overset{L.R}{=} \lim_{n\to\infty} \frac{\left(\frac{1}{1+\frac{1}{n}}\right)\cdot \cancel{\left(\frac{-1}{n^2}\right)}}{\cancel{\left(\frac{-1}{n^2}\right)}}$

$\displaystyle = 1 \Rightarrow$ by ℓCT, $\displaystyle\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right)$ diverges.

For the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \ln\left(1+\frac{1}{n}\right)$, $\quad u_n = \ln\left(1+\frac{1}{n}\right)$

1) u_n's $> 0 \quad \forall n$

2) $u_n' = \left(\frac{1}{1+\frac{1}{n}}\right)\left(\frac{-1}{n^2}\right) < 0 \quad \forall n \Longrightarrow u_n \searrow$

3) $u_n \longrightarrow \ln 1 = 0 \Longrightarrow$ by AST, $\displaystyle\sum_{n=1}^{\infty} (-1)^n \ln\left(1+\frac{1}{n}\right)$ converges conditionally.

4) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n+\sqrt{n}} - \sqrt{n}\right)$

Sol: The series of abs. values is $\displaystyle\sum_{1}^{\infty} \left(\sqrt{n+\sqrt{n}} - \sqrt{n}\right)$

Since $\displaystyle\lim_{n\to\infty} \left(\sqrt{n+\sqrt{n}} - \sqrt{n}\right) = \lim_{n\to\infty} \left(\sqrt{n+\sqrt{n}} - \sqrt{n}\right) \frac{\sqrt{n+\sqrt{n}} + \sqrt{n}}{\sqrt{n+\sqrt{n}} + \sqrt{n}}$

$\displaystyle = \lim \frac{n+\sqrt{n} - n}{\sqrt{n+\sqrt{n}} + \sqrt{n}} = \frac{1}{2} \neq 0$

so the series of abs. values is div.

For the series $\sum (-1)^n \left(\sqrt{n+\sqrt{n}} - \sqrt{n}\right)$, note that

$u_n \longrightarrow \frac{1}{2} \neq 0$ and hence $\lim a_n$ d.n.e.
so the series diverges.

Rearranging Series

We can always rearrange the terms of a *finite* sum. The same result is true for an infinite series that is absolutely convergent (see Exercise 68 for an outline of the proof).

THEOREM 17—The Rearrangement Theorem for Absolutely Convergent Series If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $b_1, b_2, \ldots, b_n, \ldots$ is any arrangement of the sequence $\{a_n\}$, then $\sum b_n$ converges absolutely and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n.$$

Summary of Tests

1. **The *n*th-Term Test:** Unless $a_n \to 0$, the series diverges.
2. **Geometric series:** $\sum ar^n$ converges if $|r| < 1$; otherwise it diverges.
3. ***p*-series:** $\sum 1/n^p$ converges if $p > 1$; otherwise it diverges.
4. **Series with nonnegative terms:** Try the Integral Test, Ratio Test, or Root Test. Try comparing to a known series with the Comparison Test or the Limit Comparison Test.
5. **Series with some negative terms:** Does $\sum |a_n|$ converge? If yes, so does $\sum a_n$ since absolute convergence implies convergence.
6. **Alternating series:** $\sum a_n$ converges if the series satisfies the conditions of the Alternating Series Test.

أمثلة

1) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\pi}{2n+1}$

sol: Firstly, note that $\cos(n\pi) = (-1)^n$, so the series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$

Consider the series of abs. values $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{2n+1}$.

Take $b_n = \frac{1}{n}$, $\sum b_n$ diverges and by LCT

$\lim \frac{u_n}{b_n} = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$ so the series

$\sum \frac{1}{2n+1}$ diverges

For the series $\sum \frac{(-1)^n}{2n+1}$, Let $u_n = \frac{1}{2n+1}$

1) u_n's > 0 $\forall n$, (2) $u_n' = \frac{-2}{(2n+1)^2} < 0 \Rightarrow u_n \searrow$

3) $u_n \to 0$

so by AST, $\sum \frac{(-1)^n}{2n+1}$ converges conditionally.

2) $$\sum_{n=1}^{\infty} \frac{(-9)^n}{10^n + 2n}$$

Sol: The series is alternating with the term

$$\sum_{n=1}^{\infty} \frac{(-1)^n \, 9^n}{10^n + 2n}$$

The series of abs. values is $\sum u_n = \sum_{n=1}^{\infty} \frac{9^n}{10^n + 2n}$

Take $b_n = \left(\frac{9}{10}\right)^n = \frac{9^n}{10^n}$, $\sum b_n$ converges G.S. $r = \frac{9}{10}$

Consider $\lim \frac{u_n}{b_n} = \lim_{n\to\infty} \frac{\cancel{9^n}}{10^n + 2n} \cdot \frac{10^n}{\cancel{9^n}} = \lim \frac{10^n}{10^n + 2n} = 1$

so by LCT, $\sum \frac{9^n}{10^n + 2n}$ converges

(ممكن الحل باستخدام DCT)

$\Rightarrow \sum_{n=1}^{\infty} \frac{(-9)^n}{10^n + 2n}$ converges abs.

3) $$\sum_{n=1}^{\infty} \frac{(-1)^n 5^{2n+1}}{n^{3n}}$$

sol: The series of abs. values is $\displaystyle\sum_{n=1}^{\infty} \frac{5^{2n+1}}{n^{3n}}$

Using root test: $\rho = \lim \sqrt[n]{a_n} = \displaystyle\lim_{n\to\infty} \frac{5^{\left(\frac{2n+1}{n}\right)}}{n^{\left(\frac{3n}{n}\right)}}$

$$= \lim_{n\to\infty} \frac{5^{2+\frac{1}{n}}}{n^3} = \lim_{n\to\infty} \frac{5^2 \sqrt[n]{5}}{n^3} = 0 < 1$$

$\Rightarrow$ the series $\displaystyle\sum_{n=1}^{\infty} \frac{5^{2n+1}}{n^{3n}}$ converges and hence

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 5^{2n+1}}{n^{3n}}$ converges abs.