Ch 11 Parametric Equations and Polar Coordinates

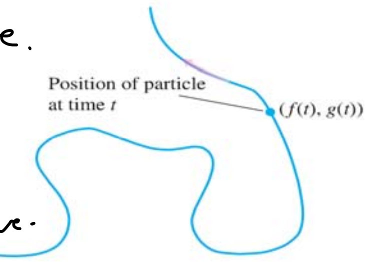
11.1 Parametrizations of Plane Curves

DEFINITION If x and y are given as functions

$$x = f(t), \qquad y = g(t)$$

over an interval I of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve**. The equations are **parametric equations** for the curve.

In this case, t is a parameter for the curve.
If $t \in I = [a,b]$ closed interval, then the point $(f(a), g(a))$ is the initial point and $(f(b), g(b))$ is the terminal point of the curve.



The equations and interval together constitute a parametrization of the curve.

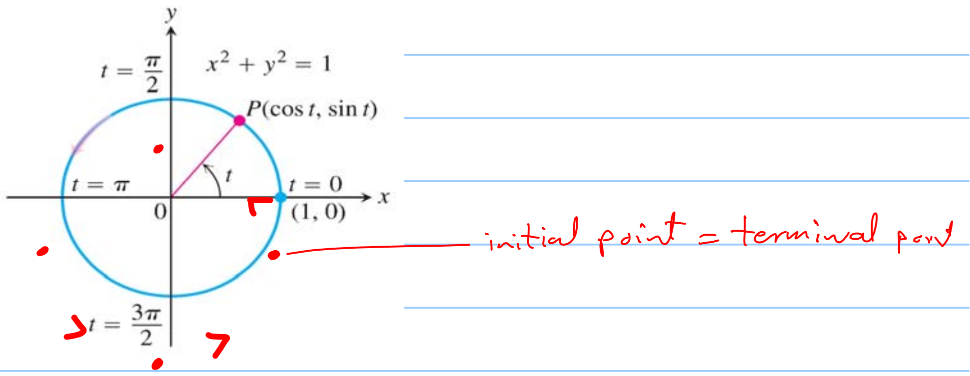
Examples: The following parametric eqs describe the position $p(x,y)$ of a particle moving in the xy-plane.

a) Identify the path traced by the particle.
b) Graph its cartesian eq on the given interval.
c) Describe the direction of the motion.

1) $x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi$

Sol: a) $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$,
which is the circle centered at the origin with radius 1.

b) At $t = 0$, $(x,y) = (\cos 0, \sin 0) = (1, 0)$. $\longrightarrow$ initial point
at $t = \pi/2$, $(x,y) = (\cos \pi/2, \sin \pi/2) = (0, 1)$,
at $t = \pi$, $(x,y) = (-1, 0)$,
at $t = 3\pi/2$, $(x,y) = (0, -1)$, and
at $t = 2\pi$, $(x,y) = (1, 0)$ $\longrightarrow$ terminal point.



c) The particle moves once on the circle $x^2+y^2=1$ counterclockwise starting at the point $P(1,0)$ and the arrow shows the direction of increasing t.

2) $x = a \sin 2t, \quad y = a \cos 2t, \quad 0 \le t \le \pi$

Sol: (a) $x^2+y^2 = a^2 \sin^2 2t + a^2 \cos^2 2t = a^2$

which is the circle centered at the origin with radius a.

(b) At $t=0$, $(x,y) = (a\sin 0, a\cos 0) = (0,a)$. $\longrightarrow$ initial point

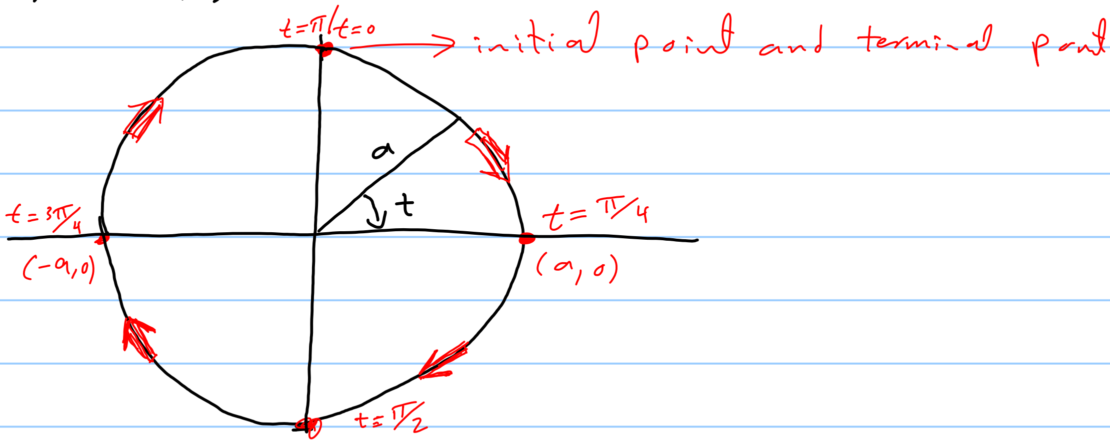
at $t=\pi/4$, $(x,y) = (a\sin \frac{\pi}{2}, a\cos\frac{\pi}{2}) = (a,0)$,

at $t=\frac{\pi}{2}$, $(x,y) = (0,-a)$,

at $t=\frac{3\pi}{4}$, $(x,y) = (-a,0)$, and

at $t=\pi$, $(x,y) = (0,a)$ $\longrightarrow$ terminal point.

(b)



c) The particle moves once on the circle $x^2+y^2=a^2$ clockwise starting at the point $P(0,1)$, and the arrow shows the direction of increasing t.

Remark: In Example 1 above, if the interval of t is $0 \le t \le \pi$, then the graph of the path is the upper half of the circle.

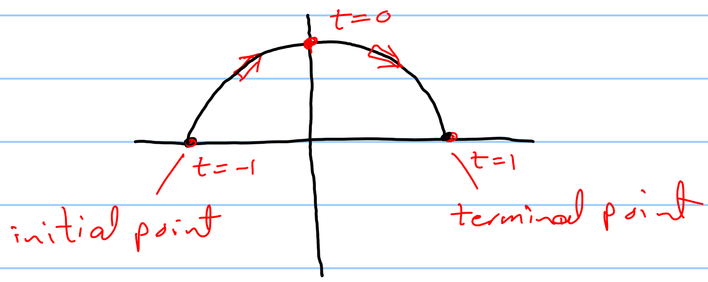
3) $x = t, \quad y = \sqrt{1-t^2}, \quad -1 \le t \le 1$

Sol: a) $y = \sqrt{1-x^2} \Rightarrow x^2+y^2=1$. which is the circle centered at the origin with radius 1.

(b) At $t=-1$: $(x,y) = (-1,0)$, $\longrightarrow$ initial point

at $t=0$: $(x,y) = (0,1)$, and

at $t=1$: $(x,y) = (1,0)$ $\longrightarrow$ terminal point

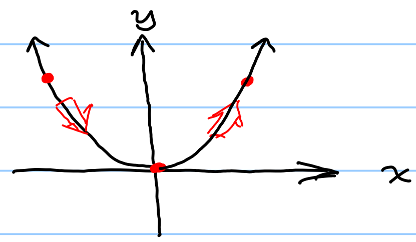


c) The particle moves on the upper half of the circle in clockwise direction.

Remark: Clearly, any curve can be represented by many different set of parametric eqs.

4) $x = 3t, \quad y = 9t^2, \quad -\infty < t < \infty.$

Sol: (a) $y = (3t)^2 \Rightarrow y = x^2$, which is parabola open up.



(b) If $t=-1$, then $(x,y) = (-3,9)$,

if $t=0$, then $(x,y) = (0,0)$,

and if $t=1$, then $(x,y) = (3,9)$,

(c) The particle moves down from the left hand side of the Parabola, passes through the origin, then moves up on the right hand side of the graph.

5) $x = 4\sin t, \quad y = 5\cos t, \quad 0 \le t \le 2\pi$

Sol: $\frac{x}{4} = \sin t, \quad \frac{y}{5} = \cos t \Rightarrow$

$$\frac{x^2}{4^2} + \frac{y^2}{5^2} = \sin^2 t + \cos^2 t = 1$$ which is eqn of

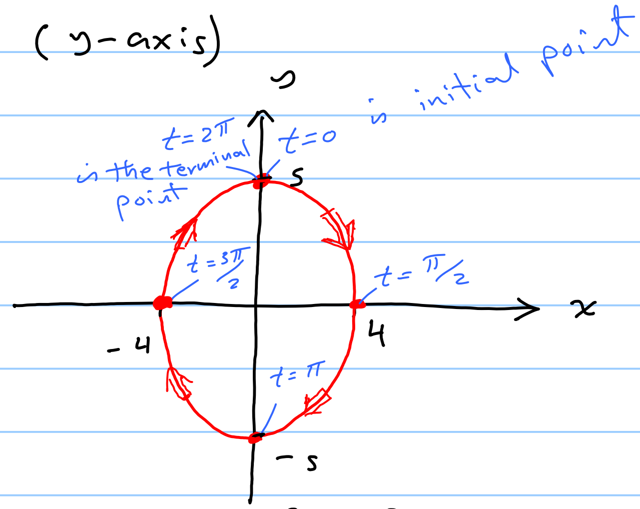
an ellipse with axis $x=0$ (y-axis)

b) At $t=0$, $(x,y) = (0,5)$,

at $t=\frac{\pi}{2}$, $(x,y) = (4,0)$

⋮

at $t = 2\pi$, $(x,y) = (0,5)$



c) The particle moves once on the ellipse $\frac{x^2}{4^2} + \frac{y^2}{5^2} = 1$ clockwise and the arrow shows the direction of increasing t.

6) The following is Example 1 in book.

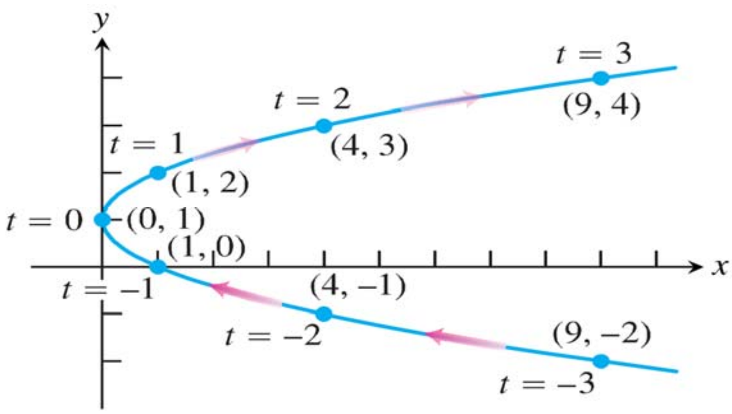


FIGURE 11.2 The curve given by the parametric equations $\boxed{x = t^2 \text{ and } y = t + 1}$ (Example 1).

$-\infty < t < \infty$

7) $x = -\sec t$, $y = \tan t$ $\quad -\frac{\pi}{2} < t < \frac{\pi}{2}$

sol: a) $x^2 - y^2 = \sec^2 t - \tan^2 t = 1$ which is the eqn of a hyperbola with axis $y=0$ (x-axis)

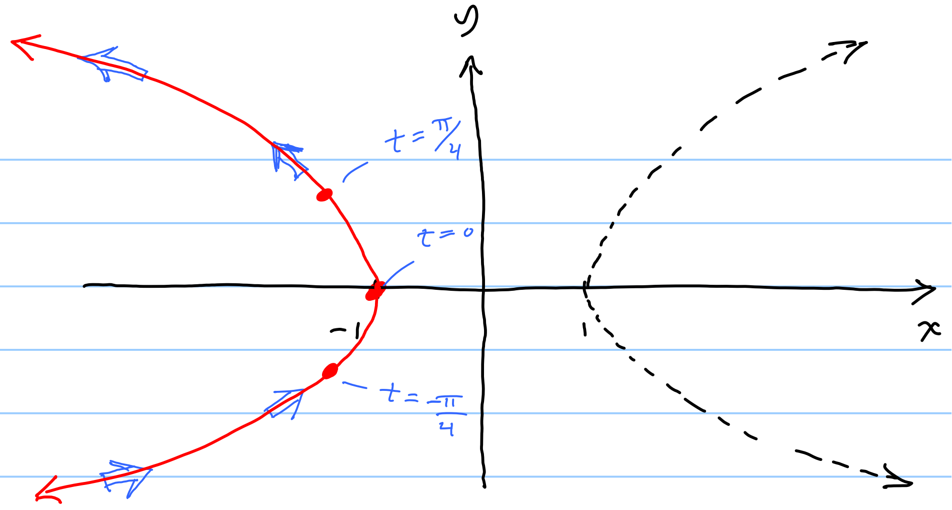
b) As $t \to -\frac{\pi}{2}$, $x \to -\infty$, $y \to -\infty$

at $t = -\frac{\pi}{4}$, $x = -\sqrt{2}$, $y = -1$

at $t = 0$, $x = -1$, $y = 0$

at $t = \frac{\pi}{4}$, $x = -\sqrt{2}$, $y = 1$

As $t \to \frac{\pi}{2}$, $x \to -\infty$, $y \to \infty$



Parametric Eqn of a Line

a) If L is a line passes through the two points (x_0, y_0) and (x_1, y_1), then we can parametrize this line as follows:

$$x = x_0 + (x_1 - x_0)t, \quad y = y_0 + (y_1 - y_0)t, \quad -\infty < t < \infty$$

b) If L is a line passes through the point (x_0, y_0) with slope m then we can parametrize this line as follows:

$$x = x_0 + t, \quad y = y_0 + mt, \quad -\infty < t < \infty$$

Example: Find two different parametrization for the line segment with endpoints $(-1, 3)$ and $(3, -2)$

Sol: لاحظ في البداية أننا بصدد إيجاد معادلة وسيطية لقطعة مستقيمة وليست للخط المستقيم كله وعليه فإننا لا نأخذ $-\infty < t < \infty$ لذا يجب إيجاد قيم t تكون نقطة البداية $(-1, 3)$ ونقطة النهاية $(3, -2)$.

باستخدام (a)

$$x = -1 + (3+1)t, \quad y = 3 + (-2-3)t \Rightarrow$$

$$x = -1 + 4t, \quad y = 3 - 5t \quad 0 \le t \le 1$$

باستخدام b $\quad m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{-5}{4}$

$\Rightarrow \quad x = -1 + t \;, \quad y = 3 - \frac{5}{4} t \qquad 0 \leq t \leq 4$

لاحظ أن قيمة $t = 4$ هي التي تجعل قيمة (x, y) مساوية لـ $(3, -2) = (x_1, y_1)$.