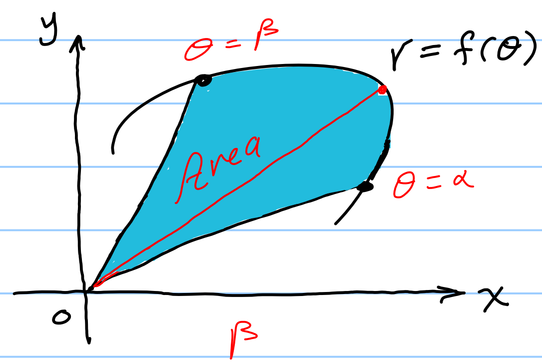
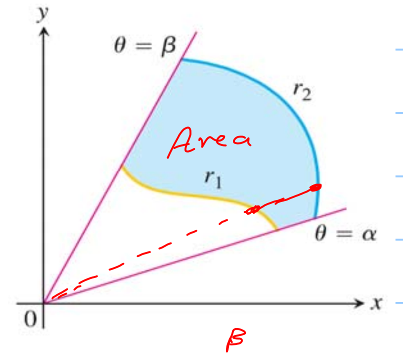
11.5 Areas and Lengths in Polar Coordinates

Note Title ٢٢/٠٦/١٨

Area in the Plane

a)


b)


$$\text{Area} = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta \qquad \text{Area} = \frac{1}{2}\int_{\alpha}^{\beta} (r_2^2 - r_1^2)\, d\theta$$

a) Area of the Fan-Shaped Region Between the Origin and the Curve $r = f(\theta), \alpha \le \theta \le \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta .$$

This is the integral of the **area differential** (Figure 11.31)

$$dA = \frac{1}{2} r^2 \, d\theta = \frac{1}{2}\left(f(\theta)\right)^2 d\theta .$$

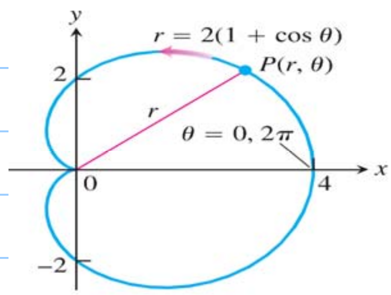
b) Area of the Region $0 \le r_1(\theta) \le r \le r_2(\theta), \alpha \le \theta \le \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r_2^2 \, d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_1^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta$$

Examples:

EXAMPLE 1 Find the area of the region in the plane enclosed by the cardioid $r = 2(1 + \cos\theta)$.

sol: $A = \int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2 \, d\theta$

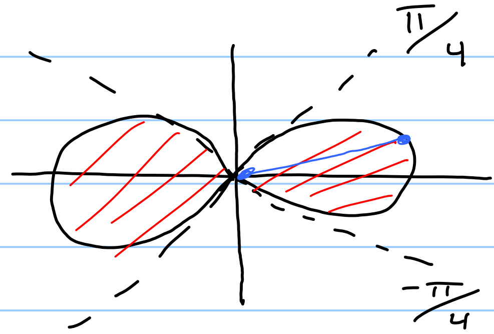


$$= \tfrac{1}{2}\cdot 4 \int_0^{2\pi} 1 + 2\cos\theta + \cos^2\theta \; d\theta = 2\int_0^{2\pi} 1 + 2\cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= 2\left[\theta + 2\sin\theta + \tfrac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\right]_0^{2\pi} = \boxed{6\pi}$$

2) Find the area of the region inside $r^2 = 4\cos 2\theta$

sol: الرسم (راجع sec 11.4)



$$A = 2\int_{-\pi/4}^{\pi/4} \tfrac{1}{2} r^2 \, d\theta$$

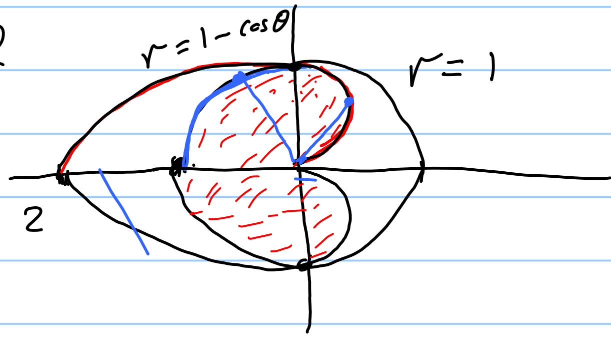
$$= \int_{-\pi/4}^{\pi/4} 4\cos 2\theta \, d\theta$$

$$= 4 \left.\frac{\sin 2\theta}{2}\right]_{-\pi/4}^{\pi/4} = 2\left[\sin\frac{\pi}{2} - \sin\frac{-\pi}{2}\right] = \boxed{4}$$

3) Find the area of:

a) the region of intersection of the circle $r=1$ and the cardioid $r = 1-\cos\theta$

sol: Firstly, Find the points of intersection



$$1 = 1 - \cos\theta \Rightarrow \cos\theta = 0$$

$$\Rightarrow \theta = -\tfrac{\pi}{2}, \tfrac{\pi}{2}.$$

تحليل المسألة : ① لوجود التماثل / يمكن ايجاد المساحة من الأعلى فقط وضربها في 2 .

② المساحة في الأعلى ليست بين منحنيين انما تنقسم إلى جزئين :

الجزء الأول : المساحة من الربع الأول وهي محصورة بين $r=0$ و $r=1-\cos\theta$
من $\theta=0$ إلى $\theta=\pi/2$.

الجزء الثاني : المساحة من الربع الثاني وهي محصورة بين $r=0$ و $r=1$
من $\theta=\pi/2$ إلى $\theta=\pi$.

$$\therefore \tfrac{1}{2}A = \int_0^{\pi/2} \tfrac{1}{2}(1-\cos\theta)^2\,d\theta + \int_{\pi/2}^{\pi} \tfrac{1}{2}\cdot 1^2\,d\theta \quad = \pi/4$$

$$= \tfrac{1}{2}\int_0^{\pi/2} 1-2\cos\theta+\cos^2\theta\,d\theta + \pi/4$$

$$= \tfrac{1}{2}\int_0^{\pi/2} 1-2\cos\theta+\tfrac{1}{2}(1+\cos 2\theta)\,d\theta + \pi/4$$

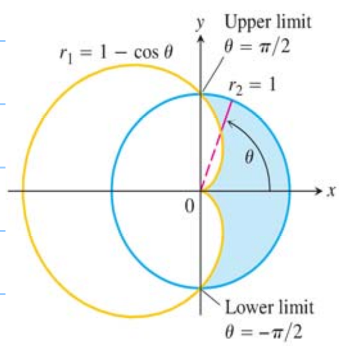
أكمل الحل :

b) the region inside the circle $r=1$ and outside the cardioid $r=1-\cos\theta$

أيضاً باستخدام التماثل يمكن ايجاد المساحة من الربع الحل:
الأول وضربها في 2 .

واضح أن $r_2=1$ أكبر من $r_1=1-\cos\theta$
في الفترة من $\theta=0$ إلى $\theta=\pi/2$

(انظر الخط الواصل من المنطقة إلى O)



$\therefore$

$$A = 2\int_0^{\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta \qquad \text{Symmetry}$$

$$= \int_0^{\pi/2} (1 - (1 - 2\cos\theta + \cos^2\theta))\, d\theta \qquad \text{Square } r_1.$$

$$= \int_0^{\pi/2} (2\cos\theta - \cos^2\theta)\, d\theta = \int_0^{\pi/2} \left(2\cos\theta - \frac{1 + \cos 2\theta}{2}\right) d\theta$$

$$= \left[2\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = 2 - \frac{\pi}{4}.$$

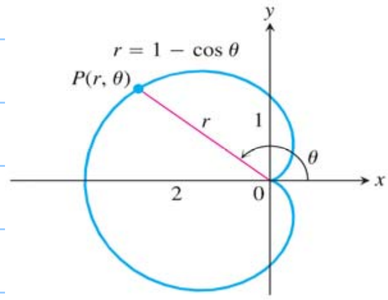
Length of a Polar Curve

If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the length of the curve is

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \tag{3}$$

Examples:

a) Find the length of the cardioid $r = 1 - \cos\theta$.



sol: $r = 1 - \cos\theta$, $\quad \frac{dr}{d\theta} = \sin\theta$

$$\therefore \; r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1-\cos\theta)^2 + \sin^2\theta$$

$$= 1 - 2\cos\theta + \cos^2\theta + \sin^2\theta = 2 - 2\cos\theta$$

$$= 4\sin^2\left(\frac{\theta}{2}\right) \qquad (\text{how??})$$

واضح انه طول المنحنى من $\theta=0$ الى $\theta=\pi$ هو $\frac{1}{2}$ الطول المطلوب.

$$\Rightarrow L = 2\int_0^{\pi} \sqrt{2 - 2\cos\theta}\, d\theta = 2\int_0^{\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta$$

$$= 4\int_0^{\pi} \left|\sin\frac{\theta}{2}\right| d\theta = 4\int_0^{\pi} \sin\frac{\theta}{2}\, d\theta \qquad \left(\sin\theta/2 > 0\right)$$

$$= 4 \cdot \left(-2\cos\frac{\theta}{2}\right)\Big|_0^{\pi} = -8\left(\cos\pi/2 - \cos 0\right) = \boxed{8}$$

b) $r = \cos^3\frac{\theta}{3}\ , \qquad 0 \le \theta \le \pi/4$

sol: $\frac{dr}{d\theta} = 3\cos^2\frac{\theta}{3} \cdot -\sin\frac{\theta}{3} \cdot \frac{1}{3} = -\sin\theta/3 \cdot \cos^2\frac{\theta}{3}$

$$\therefore\ r^2 + \left(\frac{dr}{d\theta}\right)^2 = \cos^6\frac{\theta}{3} + \sin^2\frac{\theta}{3}\cos^4\frac{\theta}{3} = \cos^4\frac{\theta}{3}\left(\cos^2\theta/3 + \sin^2\frac{\theta}{3}\right)$$

$$= \cos^4\frac{\theta}{3}$$

$$\therefore\ L = \int_0^{\pi/4} \sqrt{\cos^4\frac{\theta}{3}}\, d\theta = \int_0^{\pi/4} \left|\cos^2\frac{\theta}{3}\right| d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} 1 + \cos\frac{2\theta}{3}\, d\theta = \frac{1}{2}\left(\theta + \frac{3}{2}\sin\frac{2\theta}{3}\right)\Big|_0^{\pi/4}$$

$$= \frac{1}{2}\left[\left(\pi/4 + 3/2 \cdot \frac{1}{2}\right) - \left(0 + 3/2 \cdot 0\right)\right] = \boxed{\frac{\pi}{8} + \frac{3}{8}}$$

حلو

Example: Find the length of the curve

$$r = \sqrt{1 + \sin 2\theta}\ , \qquad 0 \le \theta \le \sqrt{2}\pi$$

sol: $\frac{dr}{d\theta} = \frac{\cos 2\theta}{\sqrt{1 + \sin 2\theta}} \Longrightarrow$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1+\sin 2\theta) + \frac{\cos^2 2\theta}{1+\sin 2\theta}$$

$$= \frac{1 + 2\sin 2\theta + \sin^2 2\theta + \cos^2 2\theta}{1+\sin 2\theta}$$

$$= \frac{2 + 2\sin 2\theta}{1+\sin 2\theta} = 2$$

$$\therefore \quad L = \int_0^{\sqrt{2}\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{2}\, d\theta = \boxed{2\pi}$$